

# Two-dimensional Brownian vortices

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## Abstract

We introduce a stochastic model of two-dimensional Brownian vortices associated with the canonical ensemble. The point vortices evolve through their usual mutual advection but they experience in addition a random velocity and a systematic drift generated by the system as a whole. The statistical equilibrium state of this stochastic model is the Gibbs canonical distribution. We consider a single species system and a system made of two types of vortices with positive and negative circulations. At positive temperatures, like-sign vortices repel each other (“plasma” case) and at negative temperatures, like-sign vortices attract each other (“gravity” case). We derive the stochastic equation satisfied by the exact vorticity field and the Fokker-Planck equation satisfied by the  $N$ -body distribution function. We present the BBGKY-like hierarchy of equations satisfied by the reduced distribution functions and close the hierarchy by considering an expansion of the solutions in powers of  $1/N$ , where  $N$  is the number of vortices, in a proper thermodynamic limit. For spatially inhomogeneous systems, we derive the kinetic equations satisfied by the smooth vorticity field in a mean field approximation valid for  $N \rightarrow +\infty$ . For spatially homogeneous systems, we study the two-body correlation function, in a Debye-Hückel approximation valid at the order  $O(1/N)$ . The results of this paper can also apply to other systems of random walkers with long-range interactions such as self-gravitating Brownian particles and bacterial populations experiencing chemotaxis. Furthermore, for positive temperatures, our study provides a kinetic derivation, from microscopic stochastic processes, of the Debye-Hückel model of electrolytes.

## 1 Introduction

Systems with long-range interactions are numerous in nature and share a lot of peculiar properties [1]. For example, it was recognized long ago that, for systems with unshielded long-range interactions, the statistical ensembles may be inequivalent at the thermodynamic limit [2, 3]. Therefore, we must carefully define the system under consideration before applying a statistical description. If we consider an *isolated* system of particles in interaction, where the microscopic dynamics is governed by Hamiltonian equations, the proper statistical ensemble is the micro-canonical ensemble. Alternatively, if we consider a *dissipative* system, where the microscopic dynamics is governed by stochastic Langevin equations, the proper statistical ensemble is the canonical ensemble. A general discussion of the dynamics and thermodynamics of Hamiltonian

and Brownian systems with long range interactions has been given in [1] and in [4, 5, 6, 7, 8] (denoted Papers I-V). To any Hamiltonian system with long range interactions, we can associate a Brownian system with long-range interactions by adding a friction force and a stochastic force in the equations of motion. For example, the self-gravitating Brownian gas studied in [9] is the canonical version of a stellar system [10] and the Brownian Mean Field (BMF) model studied in [11] is the canonical version of the Hamiltonian Mean Field (HMF) model [12]. In order to emphasize the similarities and the differences between Hamiltonian and Brownian systems, it is of great conceptual interest to study these systems in parallel.

The purpose of the present paper is to extend this study to the case of point vortices in two-dimensional hydrodynamics. Basically, a gas of point vortices constitutes a Hamiltonian system described by the Kirchhoff-Hamilton equations of motion where the coordinates  $(x, y)$  of the point vortices are canonically conjugate [13, 14]. A statistical mechanics of this system has been developed by several authors [15, 16, 17, 18, 19, 20, 21, 22, 23] following the pioneering work of Onsager [24]. For this system, the proper thermodynamic limit is such that the number of point vortices  $N \rightarrow +\infty$  while their circulation tends to zero as  $\gamma \sim 1/N$  and the area of the system remains finite  $V \sim 1$ . Assuming ergodicity, the system should reach, for  $t \rightarrow +\infty$ , a statistical equilibrium state described by the mean-field Boltzmann distribution. In order to vindicate (or not) this result and determine the timescale of the relaxation, we must develop a kinetic theory of point vortices. A general kinetic equation describing the collisional relaxation of point vortices has been derived in [25] from the Liouville equation by using a projection operator formalism. In a more recent paper [26], we have re-derived this kinetic equation from a BBGKY-like hierarchy <sup>1</sup> by using an expansion of the solutions in powers of  $1/N$  for  $N \rightarrow +\infty$ , and we have shown that this equation is valid at the order  $O(1/N)$ . In the limit  $N \rightarrow +\infty$ , the correlations between point vortices can be neglected and the kinetic equation reduces to the 2D Euler equation. This is the counterpart of the Vlasov equation in plasma physics. On the other hand, by considering an axisymmetric evolution, we have obtained an explicit expression of the collisional current at the order  $O(1/N)$  taking into account two-body correlations [25, 26]. This leads to a kinetic equation that is the counterpart of the Landau equation in plasma physics. Collective effects can be included by developing a quasilinear theory of the Klimontovich equation [28, 29] leading to a kinetic equation analogous to the Lenard-Balescu equation in plasma physics. It was numerically shown in [30] that the kinetic equation valid at the order  $O(1/N)$  does *not* in general relax towards the Boltzmann distribution because of the absence of resonances. This implies that the collisional relaxation time is larger than  $Nt_D$  (where  $t_D$  is the dynamical time). However, its precise scaling with  $N$  remains to be determined. This demands to develop a kinetic theory at the order  $1/N^2$  or smaller by taking into account three-body or higher correlations. This extension has not yet been done. Therefore, for the moment, there is no theoretical proof (coming from the kinetic theory) that the Hamiltonian point vortex gas relaxes towards the Boltzmann distribution for long times.

In this paper, we shall study a *different* model of point vortex dynamics. Our aim is to introduce a stochastic model of point vortices associated with the canonical ensemble. In other words, we wish to construct a dynamical model of point vortices that leads to the Gibbs canonical distribution at statistical equilibrium. These vortices will be called *Brownian vortices* to distinguish them from usual vortices described by Hamiltonian equations (leading to the microcanonical distribution at statistical equilibrium). By analogy with usual Brownian theory, if we want to introduce a canonical description, the first idea is to modify the usual point vortex model by adding a stochastic term in the Kirchhoff-Hamilton equations of motion. This extension has already been considered by Marchioro & Pulvirenti [31] in a different context.

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<sup>1</sup>A similar approach has been developed simultaneously in [27].

Their motivation was to take into account viscous effects in the point vortex dynamics. They considered a system of point vortices evolving through their usual mutual advection and experiencing, in addition, a random velocity whose strength is proportional to the square root of the viscosity. They showed that, in a mean-field limit valid for  $N \rightarrow +\infty$ , the evolution of the smooth vorticity field is described by the Navier-Stokes equation (instead of the 2D Euler equation). This model introduces a source of dissipation but it does not lead to the canonical distribution at statistical equilibrium (the Navier-Stokes equation leads to the trivial state  $\omega = 0$  for  $t \rightarrow +\infty$ ). If we want to obtain the canonical distribution, we need to introduce, in addition, a drift velocity produced by the vortices as a whole. This drift velocity, which is perpendicular to the advective velocity, is the counterpart of the dynamical friction in usual Brownian theory. The drift coefficient, similar to a mobility, is determined so as to yield the Boltzmann factor at statistical equilibrium. It is related to the viscosity and to the inverse temperature by a sort of Einstein relation.

This model of Brownian vortices was introduced in a Proceedings paper [32] and is here systematically developed. In a sense, this stochastic model takes into account the coupling with a thermal bath imposing the temperature rather than the energy. Therefore, it describes a system of stochastically forced vortices in contrast to the usual point vortex model describing an isolated system. At the present time, it is not clear how such a thermal bath can be realized in nature. However, this stochastic model is well-defined mathematically and constitutes *by definition* the dynamical model of point vortices associated with the canonical ensemble. Therefore, its study is interesting in its own right since it provides the out-of-equilibrium version of the situation described by the Gibbs canonical measure at equilibrium. As we said previously, it is of conceptual interest to compare the microcanonical and the canonical dynamics and study them in parallel. On the other hand, by a proper reinterpretation of the parameters, this stochastic model is isomorphic to the overdamped dynamics of Brownian particles with long-range interactions such as self-gravitating Brownian particles [9], bacterial populations experiencing chemotaxis [33], Debye-Hückel electrolytes [34] or electric charges moving in a strong vertical magnetic field (see Appendix A). Therefore, it provides an interesting model of random walkers in long-range interactions and enters in the general class of stochastic processes studied in [4, 5]. In addition, this model allows for many generalizations with respect to previous works: (i) Point vortices can have positive and negative circulations. This generalizes the studies of self-gravitating Brownian particles and bacterial populations where the mass of the particles is always positive. (ii) Since point vortices have no inertia, the temperature can be either positive or negative. At negative temperatures, point vortices of the same sign have the tendency to attract each other (like stars in a galaxy <sup>2</sup>) and at positive temperatures they have the tendency to repel each other (like electric charges in a plasma). Therefore, by changing the sign of the temperature, we can study either attractive or repulsive interactions. (iii) The equilibrium states of point vortices in the canonical ensemble have been studied in detail and present peculiar features associated with the existence of critical temperatures beyond which there is no equilibrium [36, 21, 37, 38]. Our dynamical model will allow to explore this range of parameters where rich and interesting behaviors are expected to occur. Therefore, on the viewpoint of statistical mechanics, this stochastic model of Brownian vortices has very rich properties and deserves a specific attention.

The paper is organized as follows. In Sec. 2, we consider a single species system of Brownian vortices. Starting from the Fokker-Planck equation satisfied by the  $N$ -body distribution function, we derive the BBGKY-like hierarchy for the reduced distribution functions (see Sec.

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<sup>2</sup>The numerous analogies between the statistical mechanics of 2D point vortices at negative temperatures and the statistical mechanics of stellar systems have been described in [35].

2.2). We close this hierarchy by considering an expansion of the solutions in powers of  $1/N$  for  $N \gg 1$ . For  $N \rightarrow +\infty$  (see Sec. 2.3), we derive the mean field Fokker-Planck equation satisfied by the one-body distribution function (smooth vorticity field). This is the counterpart of the Vlasov regime in plasma physics. For spatially homogeneous systems, we derive the equation satisfied by the two-body correlation function at the order  $O(1/N)$  and investigate the stability of the system (see Sec. 2.4). This is the counterpart of the Debye-Hückel theory in plasma physics. In Sec. 2.5, we derive the stochastic equation satisfied by the exact vorticity field before averaging over the noise. If we average this equation over the noise, we recover the first equation of the BBGKY hierarchy. In Secs. 2.6 and 2.7, we consider specific forms of potentials of interaction corresponding to the ordinary Newtonian (or Coulombian) potential in two dimensions and to the Yukawa (or Rossby) potential appropriate to geophysical flows. For spatially homogeneous systems, we determine the two-body correlation function and the caloric curve relating the average energy to the temperature. We specifically distinguish the case of positive and negative temperatures. In Sec. 3, we consider a multi-species gas of Brownian vortices and focus particularly on the two-species system where the vortices have positive and negative circulations. We derive the BBGKY-like hierarchy and consider the Vlasov limit  $N \rightarrow +\infty$  for the one-body distribution functions and the Debye-Hückel approximation (valid at the order  $O(1/N)$ ) for the two-body correlation functions. Finally, in Sec. 4, we conclude and discuss some perspectives of our work.

## 2 The single species system

### 2.1 The stochastic equations

The canonical distribution of a single species system of point vortices with individual circulations  $\gamma$  is

$$P_N = \frac{1}{Z} e^{-\beta H}, \quad (1)$$

where  $H = \sum_{i<j} \gamma^2 u(\mathbf{r}_i, \mathbf{r}_j) + \frac{1}{2} \gamma^2 \sum_{i=1}^N v(\mathbf{r}_i, \mathbf{r}_i)$  is the Hamiltonian,  $Z(\beta) = \int e^{-\beta H} d\mathbf{r}_1 \dots d\mathbf{r}_N$  the partition function and  $\beta = 1/T$  the inverse temperature which can be either positive or negative. The free energy is  $F(\beta) = -(1/\beta) \ln Z(\beta)$ . We have used the general Green's function formulation of Lin [39], which is valid in an arbitrary bounded domain<sup>3</sup>. For convenience, we shall note  $U \equiv \sum_{i<j} u(\mathbf{r}_i, \mathbf{r}_j) + \frac{1}{2} \sum_{i=1}^N v(\mathbf{r}_i, \mathbf{r}_i)$ . We ask: *can we formally construct a dynamical model of point vortices associated with the canonical ensemble?* The answer is yes. The dynamical model of point vortices consistent with the Gibbs canonical distribution (1) at statistical equilibrium is defined by the  $N$  coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = -\gamma \mathbf{z} \times \nabla_i U - \mu \gamma^2 \nabla_i U + \sqrt{2\nu} \mathbf{R}_i(t), \quad (2)$$

where  $i = 1, \dots, N$  label the point vortices in the system. These stochastic equations define our model of two-dimensional Brownian vortices. The first term, where  $\mathbf{z}$  is a unit vector perpendicular to the plane of motion, describes the mutual advection of the point vortices. The last term is a stochastic velocity where  $\mathbf{R}_i(t)$  is a white noise such that  $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$  and  $\langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$ , and  $\nu \geq 0$  is a diffusion coefficient which plays the role of a viscosity (see below). Finally, the middle term is a drift velocity produced by the system as a whole, where  $\mu$  plays the role of a mobility in Brownian theory. The drift velocity is

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<sup>3</sup>In [19], the Green function  $u(\mathbf{r}_i, \mathbf{r}_j)$  is noted  $-G(\mathbf{r}_i, \mathbf{r}_j)$  and the function  $v(\mathbf{r}_i, \mathbf{r}_i)$  is noted  $-g(\mathbf{r}_i, \mathbf{r}_i)$ .

perpendicular to the advective velocity. This systematic drift is necessary to compensate the effect of the stochastic velocity and yield the canonical distribution at equilibrium. This is similar to the dynamical friction in Brownian theory.

The model (2) regroups and generalizes several models of particles in interaction already introduced in the literature. For  $\mu = \nu = 0$ , we recover the usual Hamiltonian model [13] of point vortices <sup>4</sup>:

$$\frac{d\mathbf{r}_i}{dt} = -\gamma \mathbf{z} \times \nabla_i U, \quad (3)$$

which is associated with the microcanonical ensemble [24, 19, 20, 26]. For  $\mu = 0$  and  $\nu > 0$ , we recover the stochastic equations considered by Marchioro & Pulvirenti [31]:

$$\frac{d\mathbf{r}_i}{dt} = -\gamma \mathbf{z} \times \nabla_i U + \sqrt{2\nu} \mathbf{R}_i(t). \quad (4)$$

Finally, without the mutual advection (obtained formally by taking  $\mathbf{z} = \mathbf{0}$ ), the stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = -\mu\gamma^2 \nabla_i U + \sqrt{2\nu} \mathbf{R}_i(t), \quad (5)$$

describe the dynamics of Brownian particles in interaction in an overdamped limit where  $\gamma$  plays the role of the mass  $m$  and  $\nu$  plays the role of the diffusion coefficient  $D_*$  (in that case, the model is valid in  $d$  dimensions) [4, 5]. In the case of attractive (e.g., Newtonian) potentials, these stochastic equations describe, for example, a gas of self-gravitating Brownian particles [9, 40] or bacterial populations experiencing chemotaxis [41, 42, 43].

## 2.2 Ensembles average

Using the same procedure as the one developed in Paper II, we can readily write down the Fokker-Planck equation associated with the stochastic process (2). The evolution of the  $N$ -body distribution function  $P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$  is given by

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \mathbf{V}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \cdot \left( \nu \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu\gamma^2 P_N \frac{\partial U}{\partial \mathbf{r}_i} \right), \quad (6)$$

where  $\mathbf{V}_i = -\gamma \mathbf{z} \times \partial U / \partial \mathbf{r}_i$  is the total advective velocity of point vortex  $i$ . For  $\nu = \mu = 0$ , we recover the Liouville equation corresponding to a microcanonical description [26]. For  $\nu > 0$  and  $\mu \neq 0$ , the stationary solution of the Fokker-Planck equation (6) is

$$P_N = \frac{1}{Z} e^{-\frac{\mu\gamma^2}{\nu} U}. \quad (7)$$

It cancels individually the r.h.s. (Fokker-Planck term) and the l.h.s. (advective term) of Eq. (6). If we compare this expression with the Gibbs canonical distribution (1), we find that the inverse temperature is related to the mobility and to the diffusion coefficient (viscosity) by

$$\beta = \frac{\mu}{\nu}, \quad (8)$$

which is the Einstein relation in the present context. Since the vortices have no inertia, the temperature can take positive or negative values. Considering Eq. (2) with the ordinary

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<sup>4</sup>The Hamiltonian point vortex system is isomorphic to a two-dimensional guiding center plasma in which charged rods move across an external magnetic field  $\mathbf{B} = B\mathbf{z}$  with the self-consistent  $\mathbf{E} \times \mathbf{B}$  drift [15]. In this analogy, the circulation  $\gamma$  plays the role of the charge  $e$  and the stream function  $\psi$  the role of the electric potential  $\Phi$  (see Appendix A).



potential of interaction (56) or (70), we see that at positive temperatures (implying  $\mu > 0$ ) like-sign vortices repel each other and at negative temperatures (implying  $\mu < 0$ ) like-sign vortices attract each other. Let us introduce the free energy  $F = E - TS$  where  $E = \langle H \rangle = \int P_N H d\mathbf{r}_1 \dots d\mathbf{r}_N$  is the average energy and  $S = -\int P_N \ln P_N d\mathbf{r}_1 \dots d\mathbf{r}_N$  the entropy. We also consider the Massieu functional <sup>5</sup>  $J \equiv -\beta F = S - \beta E$  given by

$$J[P_N] = -\int P_N \ln P_N d\mathbf{r}_1 \dots d\mathbf{r}_N - \frac{1}{2}\beta \int P_N H d\mathbf{r}_1 \dots d\mathbf{r}_N. \quad (9)$$

We see that the Gibbs state (1) is a critical point of  $J$  with the normalization constraint:  $\delta J - \alpha \int \delta P_N d\mathbf{r}_1 \dots d\mathbf{r}_N = 0$ . Furthermore,  $\delta^2 J = -(1/2) \int (\delta P_N)^2 / P_N d\mathbf{r}_1 \dots d\mathbf{r}_N < 0$ . Therefore, if the Gibbs state exists (i.e. if the partition function exists), then it is the only maximum of the free energy functional  $J[P_N]$ . Furthermore, inserting the Gibbs state (1) in the Massieu functional (9), we recover the equilibrium value of the free energy  $J(\beta) = \ln Z(\beta)$ . On the other hand, using the Fokker-Planck equation (6) we find that

$$\dot{J} = \sum_{i=1}^N \int \frac{1}{\nu P_N} \left( \nu \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu \gamma^2 P_N \frac{\partial U}{\partial \mathbf{r}_i} \right)^2 d\mathbf{r}_1 \dots d\mathbf{r}_N. \quad (10)$$

Therefore, if  $\mu \neq 0$  and  $\nu > 0$  the Fokker-Planck equation (6) satisfies an  $H$ -theorem appropriate to the canonical ensemble:  $\dot{J} \geq 0$  and  $\dot{J} = 0$  iff  $P_N$  is the canonical distribution (1). The free energy (9) is the Lyapunov functional of the Fokker-Planck equation (6). From Lyapunov's direct method, we conclude that if  $J$  is bounded from above (or, equivalently, if the partition function exists), the Fokker-Planck equation (6) will relax, for  $t \rightarrow +\infty$ , towards the Gibbs distribution (1).

Proceeding as in Paper II, we can easily obtain the exact hierarchy of equations satisfied by the reduced distributions  $P_j(\mathbf{r}_1, \dots, \mathbf{r}_j, t)$ . To that purpose, we first note that the stochastic equation (2) takes the form of Eq. (II-146) if we make the substitutions  $m \rightarrow \gamma$ ,  $D_* \rightarrow \nu$  and

$$\frac{\partial u_{ij}}{\partial \mathbf{r}_i} \rightarrow \frac{\partial' u_{ij}}{\partial \mathbf{r}_i} \equiv \frac{1}{\mu \gamma} \mathbf{z} \times \frac{\partial u_{ij}}{\partial \mathbf{r}_i} + \frac{\partial u_{ij}}{\partial \mathbf{r}_i}, \quad (11)$$

where  $\partial' u_{ij} / \partial \mathbf{r}_i$  is a convenient notation. We also need to take into account the term  $\frac{1}{2} \sum_{i=1}^N v(\mathbf{r}_i, \mathbf{r}_i)$  in the Hamiltonian. With these substitutions, we can immediately transpose the results of Paper II to the present context. The equations for the reduced distributions of order  $j = 1, \dots, N$  are given by

$$\frac{\partial P_j}{\partial t} = \sum_{i=1}^j \frac{\partial}{\partial \mathbf{r}_i} \left[ \nu \frac{\partial P_j}{\partial \mathbf{r}_i} + \mu \gamma^2 \sum_{k=1, k \neq i}^j P_j \frac{\partial' u_{ik}}{\partial \mathbf{r}_i} + \mu \gamma^2 (N-j) \int P_{j+1} \frac{\partial' u_{i,j+1}}{\partial \mathbf{r}_i} d\mathbf{r}_{j+1} + \frac{\mu \gamma^2}{2} P_j \frac{\partial'}{\partial \mathbf{r}_i} v(\mathbf{r}_i, \mathbf{r}_i) \right] \quad (12)$$

and they form the BBGKY-like hierarchy associated with the Fokker-Planck equation (6). The equilibrium distributions are obtained by cancelling the terms in brackets and by replacing  $\partial'$  by  $\partial$  (since the steady state cancels the advective term). They satisfy the equilibrium BBGKY-like hierarchy

$$\frac{\partial P_j}{\partial \mathbf{r}_1} = -\beta \gamma^2 \sum_{k=2}^j P_j \frac{\partial u_{1,k}}{\partial \mathbf{r}_1} - \beta \gamma^2 (N-j) \int P_{j+1} \frac{\partial u_{1,j+1}}{\partial \mathbf{r}_1} d\mathbf{r}_{j+1} - \frac{\beta \gamma^2}{2} P_j \frac{\partial}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1). \quad (13)$$

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<sup>5</sup>Since the temperature can be negative in two-dimensional point vortex dynamics, it is often more convenient to use the Massieu functional (which is the direct Legendre transform of the entropy) rather than the usual free energy. By an abuse of language, we shall often refer to  $J$  as the "free energy".

They can also be obtained directly from the Gibbs canonical distribution (1) by writing

$$\frac{\partial P_N}{\partial \mathbf{r}_1} = -\beta\gamma^2 P_N \frac{\partial U}{\partial \mathbf{r}_1}, \quad (14)$$

and integrating over  $\mathbf{r}_{j+1} \dots \mathbf{r}_N$ . The exact first two equations of the BBGKY-like hierarchy are

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ \nu \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu\gamma^2 (N-1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_2 d\mathbf{r}_2 + \frac{1}{2} \mu\gamma^2 P_1 \frac{\partial'}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1) \right], \quad (15)$$

$$\begin{aligned} \frac{\partial P_2}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ \nu \frac{\partial P_2}{\partial \mathbf{r}_1} + \mu\gamma^2 P_2 \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + (N-2)\mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_3 d\mathbf{r}_3 + \frac{1}{2} \mu\gamma^2 P_2 \frac{\partial'}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1) \right] \\ + (1 \leftrightarrow 2). \end{aligned} \quad (16)$$

The next step is to decompose the reduced distributions in the form of a Mayer expansion by introducing the cumulants (see Eqs (I-14) and (I-15) of Paper I). Inserting the decomposition (I-14) in Eq. (15), we obtain

$$\begin{aligned} \frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ \nu \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu\gamma^2 (N-1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) d\mathbf{r}_2 \right. \\ \left. + \mu\gamma^2 (N-1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_2 + \frac{1}{2} \mu\gamma^2 P_1 \frac{\partial'}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1) \right], \end{aligned} \quad (17)$$

where  $P'_2$  is the two-body correlation function. Next, inserting the decomposition (I-14), (I-15) in Eq. (16) and using Eq. (17) to simplify some terms, we get

$$\begin{aligned} \frac{\partial P'_2}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ \nu \frac{\partial P'_2}{\partial \mathbf{r}_1} + \mu\gamma^2 P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + \mu\gamma^2 P'_2(\mathbf{r}_1, \mathbf{r}_2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} \right. \\ - \mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 + (N-2)\mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_1, \mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 \\ - \mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_1, \mathbf{r}_3) P_1(\mathbf{r}_2) d\mathbf{r}_3 + (N-2)\mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_2, \mathbf{r}_3) P_1(\mathbf{r}_1) d\mathbf{r}_3 \\ \left. + (N-2)\mu\gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) d\mathbf{r}_3 + \frac{1}{2} \mu\gamma^2 P'_2 \frac{\partial'}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1) \right] + (1 \leftrightarrow 2), \end{aligned} \quad (18)$$

where  $P'_3$  is the three-body correlation function. These equations are exact for all  $N$ , but the hierarchy is not closed.

We now consider the thermodynamic limit  $N \rightarrow +\infty$  in such a way that the normalized inverse temperature  $\eta \equiv \beta N \gamma^2$  remains of order unity. This corresponds to a regime of *weak coupling* since the “plasma parameter”  $\beta \gamma^2 u \sim 1/N \rightarrow 0$ . It is convenient to renormalize the parameters in such a way that the individual circulations scale like  $\gamma \sim 1/N$  (so that the total circulation  $\Gamma \sim N\gamma$  is of order unity), the inverse temperature scales like  $\beta \sim N$  and the domain area scales like  $V \sim 1$ . We also assume that  $\nu \sim 1$  so that the diffusive timescale is of order unity. The dynamical time is also of order unity since  $t_D \sim 1/\omega \sim V/\Gamma \sim 1$ . With these scalings, we have  $P_1 \sim 1$ ,  $|\mathbf{r}| \sim 1$ ,  $u \sim 1$  and  $\mu \sim \beta \sim N$ . Now, by considering the scaling of the terms appearing in the different equations of the BBGKY-like hierarchy, we find that the cumulants scale like  $P'_j \sim 1/N^{j-1}$ . In particular,  $P'_2 \sim 1/N$  and  $P'_3 \sim 1/N^2$ . Therefore, up to the order  $1/N$ , the first two equations of the BBGKY-like hierarchy take the form

$$\begin{aligned} \frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ \nu \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu\gamma^2 (N-1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) d\mathbf{r}_2 \right. \\ \left. + \mu\gamma^2 N \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_2 + \frac{1}{2} \mu\gamma^2 P_1 \frac{\partial'}{\partial \mathbf{r}_1} v(\mathbf{r}_1, \mathbf{r}_1) \right], \end{aligned} \quad (19)$$

$$\begin{aligned}
\frac{\partial P'_2}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} & \left[ \nu \frac{\partial P'_2}{\partial \mathbf{r}_1} + \mu \gamma^2 P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} \right. \\
& - \mu \gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_1(\mathbf{r}_1) P_1(\mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 + N \mu \gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_1, \mathbf{r}_2) P_1(\mathbf{r}_3) d\mathbf{r}_3 \\
& \left. + N \mu \gamma^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P'_2(\mathbf{r}_2, \mathbf{r}_3) P_1(\mathbf{r}_1) d\mathbf{r}_3 \right] + (1 \leftrightarrow 2). \tag{20}
\end{aligned}$$

The hierarchy is now closed because the three-body correlation function can be neglected at the order  $O(1/N)$ .

On the other hand, the average energy

$$E = \langle H \rangle = \sum_{i < j} \int \gamma^2 u_{ij} P_N d\mathbf{r}_1 \dots d\mathbf{r}_N + \frac{1}{2} \sum_{i=1}^N \int \gamma^2 v_{ii} P_N d\mathbf{r}_1 \dots d\mathbf{r}_N, \tag{21}$$

can be written

$$E = \frac{1}{2} N(N-1) \gamma^2 \int P_2(\mathbf{r}_1, \mathbf{r}_2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 + \frac{N}{2} \gamma^2 \int P_1(\mathbf{r}_1) v(\mathbf{r}_1, \mathbf{r}_1) d\mathbf{r}_1. \tag{22}$$

Substituting the Mayer decomposition (I-14) in the previous equation, and introducing the smooth vorticity field  $\omega = N\gamma P_1$  and the stream function

$$\psi(\mathbf{r}, t) = \int u(\mathbf{r}, \mathbf{r}') \omega(\mathbf{r}', t) d\mathbf{r}', \tag{23}$$

we obtain

$$\begin{aligned}
E = \frac{1}{2} \int \omega \psi d\mathbf{r} - N \gamma^2 \int P_1(\mathbf{r}_1) u_{12} P_1(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \\
+ \frac{1}{2} N(N-1) \gamma^2 \int P'_2(\mathbf{r}_1, \mathbf{r}_2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 + \frac{N}{2} \gamma^2 \int P_1(\mathbf{r}_1) v(\mathbf{r}_1, \mathbf{r}_1) d\mathbf{r}_1. \tag{24}
\end{aligned}$$

### 2.3 The mean field approximation

For  $N \rightarrow +\infty$ , we can neglect the two-body correlation function  $P'_2 \sim 1/N$ . In that case, the two-body distribution function factorizes in two one-body distribution functions:

$$P_2(\mathbf{r}_1, \mathbf{r}_2, t) = P_1(\mathbf{r}_1, t) P_1(\mathbf{r}_2, t) + O(1/N). \tag{25}$$

More generally, for  $N \rightarrow +\infty$ , we have the factorization

$$P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = \prod_{i=1}^N P_1(\mathbf{r}_i, t). \tag{26}$$

This is the equivalent of the mean field approximation (valid in the Vlasov regime) in plasma physics or stellar dynamics. The mean field approximation is exact at the thermodynamic limit  $N \rightarrow +\infty$ . Introducing the smooth vorticity field  $\omega = N\gamma P_1$  and the stream function (23), the first equation (19) of the BBGKY-like hierarchy reduces to the mean field Fokker-Planck equation<sup>6</sup>

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta \gamma \omega \nabla \psi) \equiv -\nabla \cdot \mathbf{J}, \tag{27}$$

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<sup>6</sup>To avoid confusions, it should be emphasized that the mean field Fokker-Planck equation (27) has a completely different origin and interpretation from the Fokker-Planck equation (140) derived in [26]. In [26], the



where  $\mathbf{u} = -\mathbf{z} \times \nabla\psi$  is the mean field velocity. The mean field Fokker-Planck equation (27) conserves the circulation  $\Gamma = N\gamma = \int \omega d\mathbf{r}$ , proportional to the vortex number. On the other hand, its steady solution is the mean field Boltzmann distribution

$$\omega(\mathbf{r}) = Ae^{-\beta\gamma\psi(\mathbf{r})}. \quad (28)$$

This distribution cancels individually the advective term (l.h.s.) and the Fokker-Planck term (r.h.s.) of Eq. (27). The mean field Boltzmann distribution can also be obtained from the equilibrium BBGKY-like hierarchy in the limit  $N \rightarrow +\infty$  (see Paper I). Substituting Eq. (28) in Eq. (23), we obtain an integro-differential equation  $\nabla \ln \omega = -\beta\gamma \int \nabla u(\mathbf{r}, \mathbf{r}')\omega(\mathbf{r}') d\mathbf{r}'$  determining the vorticity field at statistical equilibrium.

For  $N \rightarrow +\infty$ , using Eq. (26), the Boltzmann entropy  $S = -\int P_N \ln P_N d\mathbf{r}_1 \dots d\mathbf{r}_N$  reduces to

$$S = -N \int P_1(\mathbf{r}, t) \ln P_1(\mathbf{r}, t) d\mathbf{r}. \quad (29)$$

It can be written (up to an additive constant)

$$S[\omega] = - \int \frac{\omega}{\gamma} \ln \frac{\omega}{\gamma} d\mathbf{r}. \quad (30)$$

This expression of the Boltzmann entropy can also be obtained from a classical combinatorial analysis [15, 30]. On the other hand, for  $N \rightarrow +\infty$ , we obtain from Eq. (24) the mean field energy

$$E[\omega] = \frac{1}{2} \int \omega \psi d\mathbf{r}. \quad (31)$$

The mean field free energy (more precisely the mean field Massieu function) is  $J[\omega] = S[\omega] - \beta E[\omega]$  where  $S$  and  $E$  are given by Eqs. (30) and (31). It can be obtained from the free energy (9) by using the mean field approximation (26) valid for  $N \rightarrow +\infty$  [5]. The kinetic equation (27) can then be rewritten in the form

$$\frac{\partial \omega(\mathbf{r}, t)}{\partial t} - \left( \mathbf{z} \times \nabla \frac{\delta E[\omega]}{\delta \omega(\mathbf{r}, t)} \right) \cdot \nabla \omega(\mathbf{r}, t) = -\nabla \cdot \left[ \nu \gamma \omega(\mathbf{r}, t) \nabla \frac{\delta J[\omega]}{\delta \omega(\mathbf{r}, t)} \right]. \quad (32)$$

The steady state corresponds to a uniform value of  $\alpha = \delta J / \delta \omega(\mathbf{r})$  leading to the Boltzmann distribution (28). On the other hand, it is straightforward to establish that

$$\dot{J} = \int \frac{\mathbf{J}^2}{\nu \gamma \omega} d\mathbf{r} = \int \nu \gamma \omega \left( \nabla \frac{\delta J}{\delta \omega} \right)^2 d\mathbf{r} \geq 0. \quad (33)$$

Therefore, if  $\mu \neq 0$  and  $\nu > 0$  the mean field Fokker-Planck equation (27) satisfies an  $H$ -theorem appropriate to the canonical ensemble:  $\dot{J} \geq 0$  and  $\dot{J} = 0$  iff  $\omega$  is the mean field

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Fokker-Planck equation (140) describes the relaxation of a test vortex (tagged particle) in a *fixed* distribution of field vortices at statistical equilibrium (thus  $\psi(\mathbf{r})$  is a given function independent on time). Here, the mean field Fokker-Planck equation (27) describes the evolution of a Brownian system of point vortices as a whole for an assumed microscopic dynamics of the form (2) (thus  $\psi(\mathbf{r}, t)$  is a function of time produced self-consistently by the distribution of point vortices according to Eq. (23)). The mean field Fokker-Planck equation (27) is also physically different from the relaxation equation (12) with a time dependent temperature  $\beta(t)$  introduced by Robert & Sommeria [44] to phenomenologically describe the violent relaxation of the 2D Euler equation on a coarse-grained scale.

Boltzmann distribution (28). The free energy  $J[\omega]$  is the Lyapunov functional of the Fokker-Planck equation (27). The Boltzmann distribution (28) is a critical point of  $J$  at fixed  $\Gamma$  (cancelling the first variations) and it is linearly dynamically stable iff it is a (local) maximum of  $J$  at fixed  $\Gamma$ . If  $J$  is bounded from above, we conclude from Lyapunov's direct method, that the mean field Fokker-Planck equation (27) will reach, for  $t \rightarrow +\infty$ , a (local) maximum of  $J$  at fixed circulation. If several local maxima exist, the selection of the maximum will depend on a complicated notion of basin of attraction. If there is no maximum of free energy at fixed circulation, the system will have a peculiar behavior and will generate singularities associated with vortex collapse (see Secs. 2.6 and 2.7). We note that the relaxation equation (27) can be obtained from a maximum entropy production principle (MEPP) by maximizing the production of free energy  $\dot{J}$  at fixed circulation (and other physical constraints) [45, 46].

Finally, we note that the corresponding equation for the velocity field  $\mathbf{u} = -\mathbf{z} \times \nabla\psi$  is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} - \nu \beta \gamma \omega \mathbf{u}, \quad (34)$$

where  $p(\mathbf{r}, t)$  is a pressure field. Indeed, taking the curl of this equation and using  $\omega \mathbf{z} = \nabla \times \mathbf{u}$ , we recover Eq. (27) [the intermediate steps of the calculation make use of the identity of vector analysis  $\Delta \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$  which reduces to  $\Delta \mathbf{u} = -\nabla \times (\omega \mathbf{z}) = \mathbf{z} \times \nabla \omega$  for an incompressible velocity field  $\mathbf{u} = -\mathbf{z} \times \nabla \psi$ . On the other hand,  $\nabla \times (\mathbf{z} \times \nabla \omega) = \Delta \omega \mathbf{z}$  and  $\nabla \times (\omega \mathbf{u}) = \omega \nabla \times \mathbf{u} + \nabla \omega \times \mathbf{u} = -\nabla \cdot (\omega \nabla \psi) \mathbf{z}$ ; finally, the transformation of the advective term is standard]. Interestingly, we note that the systematic drift of the point vortices  $-\mu \gamma \nabla \psi$  in Eq. (27) corresponds to a friction force  $-\mu \gamma \omega \mathbf{u}$  in the equation (34) for the velocity. This “duality” is a particularity of the Brownian point vortex system which has no counterpart in systems of material particles.

## 2.4 The two-body correlation function

The evolution of the two-body correlation function at the order  $1/N$  is determined by the second equation (20) of the BBGKY hierarchy. Let us consider a spatially homogeneous distribution of point vortices (to leading order) so that  $P_1(\mathbf{r}, t) = P_0 + \hat{P}(\mathbf{r}, t)$  where  $\hat{P}$  is of order  $1/N$ . Here, we consider an infinite domain so that  $u(\mathbf{r}, \mathbf{r}') = u(|\mathbf{r} - \mathbf{r}'|)$ . Writing  $P_2'(\mathbf{r}, \mathbf{r}', t) = P_0^2 h(|\mathbf{r} - \mathbf{r}'|, t)$ , the second equation (20) of the BBGKY hierarchy equation simplifies into

$$\frac{\partial h}{\partial t} = 2\nu \Delta \left[ h + \beta \gamma^2 u + \beta n \gamma^2 \int h(\mathbf{y}, t) u(|\mathbf{x} - \mathbf{y}|) d\mathbf{y} \right], \quad (35)$$

where  $\mathbf{x} = \mathbf{r} - \mathbf{r}'$ . The steady distribution of this equation satisfies an equation of the form

$$\frac{\partial h}{\partial \mathbf{x}} = -\beta \gamma^2 \frac{\partial u}{\partial \mathbf{x}} - \beta n \gamma^2 \int h(\mathbf{y}, t) \frac{\partial u}{\partial \mathbf{x}}(|\mathbf{x} - \mathbf{y}|) d\mathbf{y}. \quad (36)$$

Noting that the integral is a convolution, this integro-differential equation is easily solved in Fourier space leading to

$$\hat{h}_{eq}(\mathbf{k}) = \frac{-\beta \gamma^2 \hat{u}(k)}{1 + (2\pi)^2 \beta n \gamma^2 \hat{u}(k)}. \quad (37)$$

The dynamical equation (35) can be solved similarly and the temporal evolution of the two-body correlation function in Fourier space with initial condition  $\hat{h}(\mathbf{k}, 0) = 0$  is found to be

$$\hat{h}(\mathbf{k}, t) = \hat{h}_{eq}(k) \left\{ 1 - e^{-2\nu k^2 [1 + (2\pi)^2 \beta n \gamma^2 \hat{u}(k)] t} \right\}. \quad (38)$$

From this equation, we see that the homogeneous equilibrium distribution is stable iff

$$1 + (2\pi)^2 \beta n \gamma^2 \hat{u}(k) > 0, \quad (39)$$

for any wave vector  $\mathbf{k}$  (otherwise, the wave vectors that do not satisfy this relation correspond to unstable modes that grow exponentially rapidly with time). This inequality usually determines a critical temperature  $\beta_0$  (similar to a spinodal point) beyond which the homogeneous system becomes unstable [4]. The stability criterion (39) can also be obtained by studying the linear dynamical stability of a spatially homogeneous steady solution of the mean field Fokker-Planck equation (27) with  $\omega = N\gamma P_0$  and  $\mathbf{u} = \mathbf{0}$ . Indeed, decomposing the perturbation in normal modes as  $\delta\omega \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\sigma t)}$ , the dispersion relation obtained by linearizing the Fokker-Planck equation (27) is  $i\sigma = \nu k^2 [1 + (2\pi)^2 \beta n \gamma^2 \hat{u}(k)]$ . The system is stable iff  $i\sigma > 0$  for all  $\mathbf{k}$  corresponding to the inequality (39). This inequality is also implied by Eq. (149) of Appendix B and by the condition that the equilibrium state must be a maximum of  $J[\omega]$  at fixed  $\Gamma[\omega]$  (see Sec. 4.4. of [4]).

## 2.5 The exact vorticity field

In this section, we shall derive the stochastic equation satisfied by the exact vorticity field before averaging over the noise. To simplify the expressions, we shall work in an infinite domain where  $u_{ij} = u(|\mathbf{r}_i - \mathbf{r}_j|)$  and  $v_{ii} = 0$ . The generalization of our results to a bounded domain is obtained by replacing  $u(|\mathbf{r}_i - \mathbf{r}_j|)$  by  $u(\mathbf{r}_i, \mathbf{r}_j)$  and by adding a term  $(1/2)\gamma^2 \nabla v(\mathbf{r}, \mathbf{r})$  in the velocity field. The exact vorticity field is expressed as a sum of Dirac distributions in the form

$$\omega_d(\mathbf{r}, t) = \gamma \sum_i^N \delta(\mathbf{r} - \mathbf{r}_i(t)). \quad (40)$$

Adapting the calculations of [47, 48, 49, 50, 8, 43] to the present context, we find that the exact vorticity field is solution of the stochastic equation

$$\begin{aligned} \frac{\partial \omega_d}{\partial t}(\mathbf{r}, t) - \mathbf{z} \times \nabla \cdot \left( \omega_d(\mathbf{r}, t) \nabla \int \omega_d(\mathbf{r}', t) u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' \right) &= \nu \Delta \omega_d(\mathbf{r}, t) \\ + \mu \gamma \nabla \cdot \left( \omega_d(\mathbf{r}, t) \nabla \int \omega_d(\mathbf{r}', t) u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' \right) &+ \nabla \cdot \left( \sqrt{2\nu\gamma\omega_d(\mathbf{r}, t)} \mathbf{R}(\mathbf{r}, t) \right), \end{aligned} \quad (41)$$

where  $\mathbf{R}(\mathbf{r}, t)$  is a Gaussian random field such that  $\langle \mathbf{R}(\mathbf{r}, t) \rangle = \mathbf{0}$  and  $\langle R^\alpha(\mathbf{r}, t) R^\beta(\mathbf{r}', t') \rangle = \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$ . Introducing the exact stream function

$$\psi_d(\mathbf{r}, t) = \int \omega_d(\mathbf{r}', t) u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}', \quad (42)$$

and the exact velocity field  $\mathbf{u}_d = -\mathbf{z} \times \nabla \psi_d$ , the stochastic equation (41) can be rewritten in the form

$$\frac{\partial \omega_d}{\partial t} + \mathbf{u}_d \cdot \nabla \omega_d = \nu \Delta \omega_d + \nabla \cdot (\mu \gamma \omega_d \nabla \psi_d) + \nabla \cdot (\sqrt{2\nu\gamma\omega_d} \mathbf{R}). \quad (43)$$

Let us consider some particular cases. For  $\nu = \mu = 0$ , we find that the exact vorticity field is solution of the 2D Euler equation

$$\frac{\partial \omega_d}{\partial t} + \mathbf{u}_d \cdot \nabla \omega_d = 0. \quad (44)$$

This is the counterpart of the Klimontovich equation in plasma physics. On the other hand, for the model considered by Marchioro & Pulvirenti [31] where  $\mu = 0$  and  $\nu > 0$ , the exact vorticity field is solution of the stochastic equation

$$\frac{\partial \omega_d}{\partial t} + \mathbf{u}_d \cdot \nabla \omega_d = \nu \Delta \omega_d + \nabla \cdot (\sqrt{2\nu\gamma\omega_d} \mathbf{R}). \quad (45)$$

This equation was not given in [31]. Introducing the exact free energy functional

$$\mathcal{J} \equiv \mathcal{S} - \beta \mathcal{H} = - \int \frac{\omega_d(\mathbf{r}, t)}{\gamma} \ln \frac{\omega_d(\mathbf{r}, t)}{\gamma} d\mathbf{r} - \frac{1}{2} \beta \int \omega_d(\mathbf{r}, t) u(|\mathbf{r} - \mathbf{r}'|) \omega_d(\mathbf{r}', t) d\mathbf{r} d\mathbf{r}', \quad (46)$$

the stochastic equation (43) can be rewritten in the form

$$\begin{aligned} & \frac{\partial \omega_d(\mathbf{r}, t)}{\partial t} - \left( \mathbf{z} \times \nabla \frac{\delta \mathcal{H}[\omega_d]}{\delta \omega_d(\mathbf{r}, t)} \right) \cdot \nabla \omega_d(\mathbf{r}, t) \\ &= -\nabla \cdot \left[ \nu \gamma \omega_d(\mathbf{r}, t) \nabla \frac{\delta \mathcal{J}[\omega_d]}{\delta \omega_d(\mathbf{r}, t)} \right] + \nabla \cdot \left[ \sqrt{2\nu\gamma\omega_d(\mathbf{r}, t)} \mathbf{R}(\mathbf{r}, t) \right]. \end{aligned} \quad (47)$$

It can be shown [8] that this stochastic equation reproduces the equilibrium two-body correlation function  $\langle \delta \omega(\mathbf{r}) \delta \omega(\mathbf{r}') \rangle$  that is related to the function  $h(\mathbf{r} - \mathbf{r}')$  determined in Sec. 2.4 according to the identity (147). Equation (47) can be viewed as a Langevin equation for the field  $\omega_d(\mathbf{r}, t)$ . Using standard methods [50], the corresponding Fokker-Planck equation for the probability  $W[\omega_d, t]$  of the field  $\omega_d$  is

$$\begin{aligned} & \frac{\partial W[\omega_d, t]}{\partial t} + \int \frac{\delta}{\delta \omega_d(\mathbf{r}, t)} \left\{ \left( \mathbf{z} \times \nabla \frac{\delta \mathcal{H}[\omega_d]}{\delta \omega_d(\mathbf{r}, t)} \right) \cdot \nabla \omega_d(\mathbf{r}, t) W[\omega_d, t] \right\} d\mathbf{r} \\ &= -\nu \gamma \int \frac{\delta}{\delta \omega_d(\mathbf{r}, t)} \left\{ \nabla \cdot \omega_d(\mathbf{r}, t) \nabla \left[ \frac{\delta}{\delta \omega_d(\mathbf{r}, t)} - \frac{\delta \mathcal{J}[\omega_d]}{\delta \omega_d(\mathbf{r}, t)} \right] W[\omega_d, t] \right\} d\mathbf{r}, \end{aligned} \quad (48)$$

and the equilibrium distribution is  $W[\omega_d] \propto e^{\mathcal{J}[\omega_d] - \alpha \omega_d}$ .

If we now average Eq. (41) over the noise, we find that the evolution of the smooth vorticity field  $\omega(\mathbf{r}, t) = \langle \omega_d \rangle$  is governed by an equation of the form

$$\begin{aligned} & \frac{\partial \omega}{\partial t}(\mathbf{r}, t) - \mathbf{z} \times \nabla \cdot \int \langle \omega_d(\mathbf{r}, t) \omega_d(\mathbf{r}', t) \rangle \nabla u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}' \\ &= \nu \Delta \omega(\mathbf{r}, t) + \mu \gamma \nabla \cdot \int \langle \omega_d(\mathbf{r}, t) \omega_d(\mathbf{r}', t) \rangle \nabla u(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}'. \end{aligned} \quad (49)$$

This equation is equivalent to Eq. (15) giving the exact evolution of the one-body distribution function. Indeed, using  $\omega(\mathbf{r}, t) = N\gamma P_1(\mathbf{r}, t)$  and the identity (see Appendix B):

$$\langle \omega_d(\mathbf{r}, t) \omega_d(\mathbf{r}', t) \rangle = N\gamma^2 P_1(\mathbf{r}, t) \delta(\mathbf{r} - \mathbf{r}') + N(N-1)\gamma^2 P_2(\mathbf{r}, \mathbf{r}', t), \quad (50)$$

we find that Eqs. (49) and (15) coincide. Furthermore, if we make the mean field approximation  $\langle \omega_d(\mathbf{r}, t) \omega_d(\mathbf{r}', t) \rangle = \omega(\mathbf{r}, t) \omega(\mathbf{r}', t)$  valid for  $N \rightarrow +\infty$ , we obtain

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nabla \cdot (\nu \nabla \omega + \mu \gamma \omega \nabla \psi), \quad (51)$$

which is equivalent to Eq. (27). In Fourier space, the mean field Fokker-Planck equation (51) can be written

$$\begin{aligned} & \frac{\partial \hat{\omega}}{\partial t}(\mathbf{k}, t) + (2\pi)^2 \int \mathbf{k} \cdot \mathbf{k}'_{\perp} \hat{\omega}(\mathbf{k} - \mathbf{k}', t) \hat{u}(\mathbf{k}') \hat{\omega}(\mathbf{k}', t) d\mathbf{k}' \\ &= -\nu k^2 \hat{\omega}(\mathbf{k}, t) - (2\pi)^2 \mu \gamma \int \mathbf{k} \cdot \mathbf{k}' \hat{\omega}(\mathbf{k} - \mathbf{k}', t) \hat{u}(\mathbf{k}') \hat{\omega}(\mathbf{k}', t) d\mathbf{k}', \end{aligned} \quad (52)$$

where we have noted  $\mathbf{k}_\perp = \mathbf{z} \times \mathbf{k}$ . Let us consider some particular cases of Eq. (51). For  $\nu = \mu = 0$ , we find that the smooth vorticity field is solution of the 2D Euler equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0. \quad (53)$$

This is the counterpart of the Vlasov equation in plasma physics. On the other hand, for the model considered by Marchioro & Pulvirenti [31] where  $\mu = 0$  and  $\nu > 0$ , the smooth vorticity field is solution of the Navier-Stokes equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega. \quad (54)$$

The derivation of this mean field equation was proven rigorously in [31].

Equation (51) for the ensemble averaged vorticity field  $\omega(\mathbf{r}, t)$  is a deterministic equation since we have averaged over the noise. In contrast, Eq. (41) for the exact vorticity field  $\omega_d(\mathbf{r}, t)$  is a stochastic equation taking into account fluctuations. However, it is not very useful in practice since the field  $\omega_d(\mathbf{r}, t)$  is a sum of Dirac distributions, not a regular function. Therefore, it is easier to directly solve the stochastic equations (2) rather than the equivalent Eq. (41). Following [50, 8], we can keep track of fluctuations while avoiding the problem of  $\delta$ -functions by defining a “coarse-grained” vorticity field  $\bar{\omega}(\mathbf{r}, t)$  obtained by averaging the exact vorticity field on a spatio-temporal window of finite resolution. For a weak long-range potential of interaction and for a sufficiently small spatio-temporal window, we propose to make the approximation  $\bar{\omega}^{(2)}(\mathbf{r}, \mathbf{r}', t) \simeq \bar{\omega}(\mathbf{r}, t) \bar{\omega}(\mathbf{r}', t)$ . Then, we find that the “coarse-grained” vorticity field satisfies a stochastic equation of the form

$$\frac{\partial \bar{\omega}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\omega} = \nu \Delta \bar{\omega} + \nabla \cdot (\mu \gamma \bar{\omega} \nabla \bar{\psi}) + \nabla \cdot (\sqrt{2\nu \gamma \bar{\omega}} \mathbf{R}). \quad (55)$$

If we ignore the noise term, Eq. (55) reduces to Eq. (27). In that case, the system tends to a steady state that is a maximum (global or local) of the free energy functional  $J[\bar{\omega}]$  at fixed circulation (minima or saddle points of free energy are linearly dynamically unstable). If the free energy admits several local maxima (metastable states), the selection of the steady state will depend on a notion of *basin of attraction*. Without noise, the system remains on a maximum of free energy forever. Now, in the presence of noise, the fluctuations can induce *dynamical phase transitions* from one maximum to the other. Thus, accounting correctly for fluctuations is very important when there exists metastable states (see discussion in Sec. 2.3 of [8]). This would be an interesting effect to study in more detail.

## 2.6 Newtonian or Coulombian potential

Let us make the previous results more explicit by considering particular forms of potentials of interaction. The usual potential of interaction between point vortices is solution of the Poisson equation

$$\Delta u = -\delta(\mathbf{x}). \quad (56)$$

It corresponds to a Newtonian (or Coulombian) potential in two dimensions. For box-confined vortices interacting through this potential, it can be shown that the Gibbs canonical distribution exists (i.e. the partition function is finite) if, and only, if [21, 37, 38]

$$\beta > \beta_c \equiv -\frac{8\pi}{N\gamma^2}. \quad (57)$$

At positive temperatures  $\beta > 0$ , point vortices of the same sign tend to “repel” each other and accumulate on the boundary of the domain. This corresponds to a repulsive interaction like between electric charges of the same sign in plasma physics. At negative temperatures  $\beta < 0$ , point vortices of the same sign tend to “attract” each other and form clusters. This corresponds to an attractive interaction like between stars in a galaxy. As the inverse temperature is reduced the cluster is more and more condensed and for  $\beta = \beta_c$ , the equilibrium distribution is a Dirac peak containing all the vortices. For  $\beta < \beta_c$ , there is no equilibrium state (the Gibbs canonical distribution (1) is not normalizable and the free energy (9) has no maximum). This regime can be studied dynamically with the stochastic model (2) introduced here.

In the mean field approximation valid for  $N \rightarrow +\infty$ , the evolution of the smooth vorticity is described by the coupled system

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta \gamma \omega \nabla \psi), \quad (58)$$

$$-\Delta \psi = \omega. \quad (59)$$

The equilibrium state is obtained by solving the Boltzmann-Poisson equation [35]. By a proper reinterpretation of the parameters, these equations are isomorphic to the 2D one-component Smoluchowski-Poisson (SP) system, describing a gas of self-gravitating Brownian particles [9], with an additional advective term. Like in gravity, we expect that for  $\beta \leq \beta_c$ , this system will describe the collapse of the Brownian vortex gas. In fact, if we consider an axisymmetric evolution, the advective term cancels out in Eq. (58) and we can immediately transpose the results obtained in gravity for the SP system: (i) For  $\beta > \beta_c$ , the system (58)-(59) tends to an equilibrium state in a bounded domain [52] and evaporates in an infinite domain [53]. (ii) For  $\beta = \beta_c$ , the system forms, in an infinite time, a Dirac peak containing the  $N$  point vortices. The central vorticity diverges exponentially rapidly with time in a bounded domain [52] and logarithmically in an unbounded domain [53]. (iii) For  $\beta < \beta_c$ , the system forms, in a finite time  $t_{coll}$ , a Dirac peak containing  $(\beta_c/\beta)N$  point vortices surrounded by a halo of vortices evolving pseudo self-similarly [52]. A Dirac peak containing all the point vortices is formed in the post-collapse regime  $t > t_{coll}$  in a finite time  $t_{end}$  [54].

Let us now determine the two-body correlation function of an infinite and homogeneous distribution of point vortices using Eq. (36). In fact, for  $\beta \neq 0$ , a single species system of point vortices with potential of interaction (56) cannot be spatially homogeneous. However, our approach can be justified by the following arguments: (i) We can proceed *as if* the system were infinite and homogeneous, making the equivalent of the “Jeans swindle” in astrophysics [55]. This can give us some hints about the behaviour of the correlation functions for our system. (ii) We can add a neutralizing background, like in the Jellium model of plasma physics, so that spatially homogeneous equilibrium states of point vortices now exist. (iii) We can consider a two-species system and focus on one of the two species. It will be shown in Sec. 3 that the correlation functions of a two-components, globally neutral, homogeneous system of point vortices are the *same* as those derived here by making the “Jeans swindle”. This remark justifies a posteriori the validity of the following results for a neutral system of point vortices.

In an infinite domain, the potential of interaction and its Fourier transform are given by

$$u(x) = -\frac{1}{2\pi} \ln(x), \quad (2\pi)^2 \hat{u}(k) = \frac{1}{k^2}. \quad (60)$$

Taking the divergence of Eq. (36) and using the Poisson equation (56), we find that the two-body correlation function satisfies an equation of the form

$$\Delta h - \beta n \gamma^2 h = \beta \gamma^2 \delta(\mathbf{x}). \quad (61)$$



The Fourier transform of the correlation function is

$$n(2\pi)^2 \hat{h}(k) = \frac{-\beta n \gamma^2}{k^2 + \beta n \gamma^2}. \quad (62)$$

According to the criterion (39), the system is stable for  $\beta > 0$  and unstable for  $\beta < 0$ . At positive temperatures, a point vortex of given circulation has the tendency to be surrounded by point vortices of opposite circulation (in the two-species gas). This is similar to Debye shielding in plasma physics and this allows the existence of stable homogeneous states. At negative temperatures, an infinite and homogeneous distribution of point vortices has the tendency to collapse and form clusters. This is similar to the Jeans instability in self-gravitating systems. Let us consider these two cases consecutively.

At positive temperatures, it is convenient to introduce the Debye wavenumber  $k_D = (\beta n \gamma^2)^{1/2}$ . The two-body correlation function can then be expressed as

$$n(2\pi)^2 \hat{h}(k) = \frac{-k_D^2}{k^2 + k_D^2}, \quad h(x) = -\frac{\beta \gamma^2}{2\pi} K_0(k_D x). \quad (63)$$

This corresponds to the two-dimensional Debye-Hückel theory. The correlation length is equal to the Debye length

$$L_D = \frac{1}{k_D} = \frac{1}{(\beta n \gamma^2)^{1/2}}. \quad (64)$$

The energy of interaction (correlational energy) is given by

$$E = \frac{1}{2} n^2 \gamma^2 V \int h(\mathbf{x}) u(\mathbf{x}) d\mathbf{x}, \quad \text{or} \quad E = \frac{1}{2} n^2 \gamma^2 V (2\pi)^2 \int \hat{h}(\mathbf{k}) \hat{u}(\mathbf{k}) d\mathbf{k}. \quad (65)$$

Evaluating the first integral in Eq. (65), which is convergent, we get

$$E = -\frac{N \gamma^2}{8\pi} \left[ \ln \left( \frac{\beta N \gamma^2}{4} \right) + 2\gamma_E \right], \quad \beta = \frac{4}{N \gamma^2} e^{-\frac{8\pi E}{N \gamma^2} - 2\gamma_E}, \quad (66)$$

where  $\gamma_E = 0.577\dots$  is Euler's constant <sup>7</sup>. On the other hand, using Eq. (63), the spatial correlations of the vorticity in position space and its Fourier spectrum are given by (see Appendix B):

$$\langle \delta\omega(\mathbf{r}) \delta\omega(\mathbf{r}') \rangle = n \gamma^2 \left\{ \delta(\mathbf{r} - \mathbf{r}') - \frac{\beta n \gamma^2}{2\pi} K_0(k_D |\mathbf{r} - \mathbf{r}'|) \right\}, \quad (67)$$

$$\langle \delta\hat{\omega}_{\mathbf{k}} \delta\hat{\omega}_{\mathbf{k}'} \rangle = \frac{n \gamma^2}{(2\pi)^2} \frac{1}{1 + \frac{n \beta \gamma^2}{k^2}} \delta(\mathbf{k} + \mathbf{k}'). \quad (68)$$

The spectrum is depressed at small  $k$  (i.e. long-wave fluctuations are reduced) corresponding to Debye shielding in plasma physics. The velocity distribution  $W(\mathbf{V})$  and the spatial correlations  $\langle \mathbf{V}(0) \cdot \mathbf{V}(\mathbf{r}) \rangle$  of the velocity field created by the point vortices are discussed in [56].

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<sup>7</sup>The second integral in Eq. (65) is divergent. If we introduce a lower cut-off  $k_{min} \sim V^{-1/2}$  taking into account the large but finite extent of the system, we get  $E = -n^2 \gamma^4 \beta V / (4\pi) \int_{k_{min}}^{+\infty} (1/k^2) (1/(k^2 + k_D^2)) k dk$ . This leads to the following relation  $E = -N \gamma^2 / (8\pi) \ln(1 + \beta N \gamma^2 / 8\pi)$  or  $\beta = 8\pi / (N \gamma^2) (e^{-8\pi E / (N \gamma^2)} - 1)$  between the average energy and the inverse temperature. For convenience, we have taken  $k_{max} = (8\pi/V)^{1/2}$  in order to recover the critical temperature  $\beta_c = -8\pi / (N \gamma^2)$  for  $E \rightarrow +\infty$ . However, we recall that the above results are only valid for  $\beta > 0$ .

At negative temperatures, we can introduce the equivalent the Jeans wavenumber  $k_J = (-\beta n \gamma^2)^{1/2}$ . The two-body correlation function can then be expressed as

$$n(2\pi)^2 \hat{h}(k) = \frac{k_J^2}{k^2 - k_J^2}, \quad h(x) = \frac{\beta \gamma^2}{4} Y_0(k_J x). \quad (69)$$

We note that  $h(x) = -\beta \gamma^2 u(x)$  for  $x \rightarrow 0$ . The oscillatory behaviour of the correlation function  $h(x)$ , or the fact that the denominator of  $\hat{h}(k)$  is negative for  $k < k_J$ , is related to the instability (collapse) of a homogeneous distribution of point vortices at negative temperatures (see criterion (39)). The system is unstable to large wavelengths, above the Jeans length  $L_J \sim k_J^{-1}$ . When the system becomes spatially inhomogeneous, the dynamics of clustering can be studied with the mean field equations (58)-(59).

## 2.7 Yukawa or Rossby potential

We now assume that the potential of interaction is solution of an equation of the form

$$\Delta u - k_0^2 u = -\delta(\mathbf{x}), \quad (70)$$

where  $k_0^{-1}$  is the characteristic range of the potential. This screened Poisson equation describes geophysical flows in the so-called Quasi-Geostrophic (QG) approximation. In that context,  $k_0$  is the Rossby wavenumber. The screened Poisson equation (70) also arises in the Hasegawa-Mima model [57] which describes low-frequency drift waves in magnetized plasmas. In an infinite domain, the potential of interaction and its Fourier transform are given by

$$u(x) = \frac{1}{2\pi} K_0(k_0 x), \quad (2\pi)^2 \hat{u}(k) = \frac{1}{k^2 + k_0^2}. \quad (71)$$

For a spatially homogeneous system, using Eqs. (37) and (71), the Fourier transform of the correlation function is

$$n(2\pi)^2 \hat{h}(k) = \frac{-\beta n \gamma^2}{k^2 + k_0^2 + \beta n \gamma^2}. \quad (72)$$

According to the stability criterion (39), the system is stable iff

$$\beta > -\frac{k_0^2}{n \gamma^2} \equiv \beta_0. \quad (73)$$

We note that, for a screened Coulombian potential, stable homogeneous distributions exist at negative temperatures, above a critical temperature  $\beta_0$  depending on the characteristic range  $k_0^{-1}$  of the potential. In terms of this critical temperature, the correlation function (72) can be rewritten

$$(2\pi)^2 \hat{h}(k) = \frac{-\beta \gamma^2}{k^2 + k_0^2(1 - \beta/\beta_0)}, \quad h(x) = -\frac{\beta \gamma^2}{2\pi} K_0[k_0(1 - \beta/\beta_0)^{1/2} x]. \quad (74)$$

The correlation length is

$$L = \frac{1}{k_0} (1 - \beta/\beta_0)^{-1/2}. \quad (75)$$

Let us first consider the case of positive temperatures. Introducing the Debye wavenumber  $k_D = (\beta n \gamma^2)^{1/2}$ , the correlation function and the correlation length can be written

$$n(2\pi)^2 \hat{h}(k) = \frac{-k_D^2}{k^2 + k_0^2 + k_D^2}, \quad L = \frac{1}{\sqrt{k_0^2 + k_D^2}}. \quad (76)$$

At positive temperatures, the correlation length is *smaller* than the Rossby length:  $L \leq L_0$  where  $L_0 = 1/k_0$ . This corresponds to the formation of a *screening cloud* of opposite sign vortices around each vortex due to collective effects (like in plasma physics). For  $T \rightarrow 0^+$ , i.e.  $\beta \rightarrow +\infty$ , the screening length coincides with the Debye length:  $L = L_D \rightarrow 0$ . For  $T \rightarrow +\infty$ , i.e.  $\beta \rightarrow 0^+$ , collective effects are negligible and we have  $h(\mathbf{x}) = -\beta\gamma^2 u(\mathbf{x})$  [4] so that the correlation length coincides with the Rossby length:  $L = L_0$ . At negative temperatures, introducing the Jeans wavenumber  $k_J = (-\beta n \gamma^2)^{1/2}$ , the correlation function and the correlation length can be written

$$n(2\pi)^2 \hat{h}(k) = \frac{k_J^2}{k^2 + k_0^2 - k_J^2}, \quad L = \frac{1}{\sqrt{k_0^2 - k_J^2}}. \quad (77)$$

At negative temperatures, the correlation length is *larger* than the Rossby length:  $L \geq L_0$ . This regime corresponds to the formation of a *reinforcing cloud* due to the clustering of point vortices of the same sign (like in gravity). For  $T \rightarrow -\infty$ , i.e.  $\beta \rightarrow 0^-$ , collective effects are negligible so that the correlation length coincides with the Rossby length:  $L = L_0$ . For  $T \rightarrow T_0$ , i.e.  $\beta \rightarrow \beta_0$ , the correlation length diverges. For  $\beta < \beta_0$ , the system is unstable to wavenumbers

$$k < k_{max} \equiv k_0(\beta/\beta_0 - 1)^{1/2}. \quad (78)$$

For  $\beta = \beta_0$ , we have  $k_{max} = 0$  and for  $\beta \rightarrow -\infty$ , we have  $k_{max} \simeq k_J \rightarrow +\infty$ . For a screened potential, since  $k_{max}^2 = k_J^2 - k_0^2$ , the lengthscale at which the system becomes unstable is larger than the Jeans length.

For a screened potential, the energy of interaction (65) is given by

$$E = -\frac{n^2 \gamma^4 \beta V}{4\pi} \int_0^{+\infty} \frac{1}{k^2 + k_0^2} \frac{1}{k^2 + k_0^2(1 - \beta/\beta_0)} k dk, \quad (79)$$

yielding

$$E = -\frac{N\gamma^2}{8\pi} \ln(1 - \beta/\beta_0), \quad \frac{\beta}{\beta_0} = 1 - e^{-\frac{8\pi E}{N\gamma^2}}. \quad (80)$$

On the other hand, using Eq. (74), the spatial correlations of the vorticity in position space and its Fourier spectrum are given by (see Appendix B):

$$\langle \delta\omega(\mathbf{r})\delta\omega(\mathbf{r}') \rangle = n\gamma^2 \left\{ \delta(\mathbf{r} - \mathbf{r}') - \frac{\beta n \gamma^2}{2\pi} K_0[k_0(1 - \beta/\beta_0)^{1/2}|\mathbf{r} - \mathbf{r}'|] \right\}, \quad (81)$$

$$\langle \delta\hat{\omega}_{\mathbf{k}}\delta\hat{\omega}_{\mathbf{k}'} \rangle = \frac{n\gamma^2}{(2\pi)^2} \frac{1}{1 + \frac{n\beta\gamma^2}{k^2 + k_0^2}} \delta(\mathbf{k} + \mathbf{k}'). \quad (82)$$

Interestingly, these results can also be obtained directly from the density of states (for large negative energies) [17, 19] or from the partition function [51] by making the random phase approximation. The complementary approach presented here, based on an equilibrium BBGKY-like hierarchy, is simpler because it avoids the use of complex analysis. In the present context of Brownian vortices (canonical ensemble), the control parameter is the temperature  $\beta$  while for usual vortices (microcanonical ensemble) [17, 19], the control parameter is the energy  $E$ . Let us interpret the results (81)-(82) in terms of the temperature making a parallel with the discussion given by [17] in terms of the energy. When  $T = \infty$  (i.e.  $\beta = 0$ ), the spectrum (82) is flat as for a random distribution of vortices. When  $\beta > 0$ , the spectrum is depressed at small  $k$  (i.e.

long-wave fluctuations are reduced). This corresponds to Debye shielding in plasma physics in which each particle is surrounded by a cloud of opposite sign particles. When  $\beta < 0$ , the spectrum is enhanced at small  $k$  (i.e. long-wave fluctuations are increased). This corresponds to a form of anti-shielding in which each particle is surrounded by a cloud of similar particles. This is similar to the Jeans clustering in astrophysics. The second term in the correlation function (81) arises from the vortex cloud which, on average, surrounds each particle. The radius of the cloud is  $L \sim k_0^{-1}(1 - \beta/\beta_0)^{-1/2}$  and the effective number of vortices contained in the cloud (obtained by integrating (81)) is  $N_{eff} \sim (1 - \beta_0/\beta)^{-1}$ . For  $\beta > 0$ , the cloud is of opposite circulation to the vortex and shields its effect. For  $\beta < 0$ , the cloud is of same circulation as the vortex and enhances its effect. The effective number of vortices in the cloud increases as the inverse temperature decreases and it diverges for  $\beta = \beta_0$ . This corresponds to the formation of large “clumps” or “clusters” of vortices. For  $\beta < \beta_0$ , the statistically homogeneous distribution of vortices becomes unstable and forms clusters. When the system becomes spatially inhomogeneous, the dynamics of clustering can be studied with the mean field equation (58) coupled to the screened Poisson equation  $\Delta\psi - k_0^2\psi = -\omega$ .

## 2.8 The different regimes and the orbit-averaged-Fokker-Planck equation

Using the previous kinetic theory, we can qualitatively discuss the different regimes of the dynamics of 2D Brownian vortices depending on the values of  $N$ ,  $\nu$  and  $\mu = \beta\nu$  (a similar discussion applies to self-gravitating Brownian particles [40] and to the BMF model [58]). For  $\nu = \mu = 0$  and  $N \rightarrow +\infty$ , Eq. (27) reduces to the 2D Euler equation. Starting from an unstable initial condition, the 2D Euler equation can undergo a process of violent relaxation and form a quasi-stationary state (QSS) on a short dynamical timescale  $\sim t_D$  [59, 60]. Then, finite  $N$  effects are expected to drive the “collisional” relaxation of the point vortex gas towards the microcanonical Boltzmann distribution [with temperature  $\beta(E)$ ] on a longer timescale  $t_{relax}(N)$  whose scaling with  $N$  is not yet known (see discussion in [26, 30]). For  $\nu > 0$ ,  $\mu \neq 0$  and  $N \gg 1$ , the mean field Fokker-Planck equation (27) drives the system towards the canonical Boltzmann distribution (28) [with bath temperature  $\beta$ ] on a diffusive timescale  $t_{diff} \sim R^2/\nu$  independent on  $N$ . We can distinguish three cases: (i) When  $t_D \ll t_{relax}(N) \ll t_{diff}$  (corresponding to  $N$  finite and  $\nu, \mu \rightarrow 0$ ), the system undergoes violent relaxation towards an Euler QSS, then evolves towards a *microcanonical quasi stationary state* [58] due to “collisions” (i.e. correlations due to finite  $N$  effects) and finally relaxes towards the canonical equilibrium state. (ii) When  $t_D \ll t_{diff} \ll t_{relax}(N)$  (corresponding to  $N \rightarrow \infty$  and relatively small viscosity), the system undergoes violent relaxation towards an Euler QSS, then relaxes towards the canonical equilibrium state without forming a microcanonical quasi stationary state. (iii) When  $t_{diff} \ll t_D \ll t_{relax}(N)$  (corresponding to large viscosity  $\nu, \mu \rightarrow +\infty$ ), the system undergoes a diffusive relaxation towards the canonical equilibrium state without forming a QSS.

Let us consider more specifically the second case. We assume that  $N \rightarrow +\infty$  so that “collisional” effects are completely negligible. We also assume that  $\nu \rightarrow 0$  and  $\mu \rightarrow 0$  (while  $\beta \sim 1$ ). To a first approximation, the kinetic equation (27) reduces to the 2D Euler equation. We assume that the system has reached a QSS as a result of violent relaxation. This is a stable stationary solution of the 2D Euler equation of the form  $\omega = f(\psi)$ , which depends only on the stream function. It is reached on a few dynamical times  $t_D$ . If  $\nu$  is finite, the system will evolve on a longer timescale due to the terms of drift and diffusion present in the Fokker-Planck current. If  $\nu$  and  $\mu$  are sufficiently small, the solution  $\omega(\mathbf{r}, t)$  will remain close, at any time, to a steady solution of the 2D Euler equation of the form  $\omega = \omega(\psi, t)$  that slowly evolves in time

due to the effects of diffusion and drift. Noting that

$$\frac{\partial}{\partial t}\omega(\psi(\mathbf{r}, t), t) = \frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial t}\frac{\partial\omega}{\partial\psi}, \quad (83)$$

we can rewrite the kinetic equation (27) in the form

$$\frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial t}\frac{\partial\omega}{\partial\psi} = Q(\omega), \quad (84)$$

where  $Q(\omega) = -\nabla \cdot \mathbf{J}$  is the Fokker-Planck term. Using the fact that  $\omega$  only depends on the stream function in a first approximation, we have

$$\begin{aligned} Q(\omega) &\equiv \nu \nabla \cdot (\nabla\omega + \beta\gamma\omega\nabla\psi) = \nu \nabla \cdot \left[ \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) \nabla\psi \right] \\ &= \nu \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) \Delta\psi + \nu \nabla\psi \cdot \nabla \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) \\ &= \nu \Delta\psi \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) + \nu (\nabla\psi)^2 \frac{\partial}{\partial\psi} \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right), \end{aligned} \quad (85)$$

where  $\Delta\psi = k_0^2\psi - \omega$  for the QG model ( $\Delta\psi = -\omega$  for the ordinary point vortex model). The next step is to get rid of the dependence on  $\mathbf{r}$  in Eq. (85) by averaging over the orbits. Then, the dynamical equation for  $\omega(\psi, t)$  becomes

$$\frac{\partial\omega}{\partial t} + \left\langle \frac{\partial\psi}{\partial t} \right\rangle \frac{\partial\omega}{\partial\psi} = \nu(k_0^2\psi - \omega) \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) + \nu \langle (\nabla\psi)^2 \rangle \frac{\partial}{\partial\psi} \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right), \quad (86)$$

where  $\langle X \rangle$  denotes the average of  $X(\mathbf{r}, t)$  on the elementary surface of the plane between  $\psi$  and  $\psi + d\psi$ . Defining

$$\chi(\psi, t) = \langle (\nabla\psi)^2 \rangle = \frac{\int_{\psi \leq \psi(\mathbf{r}, t) \leq \psi + d\psi} (\nabla\psi)^2 d\mathbf{r}}{\int_{\psi \leq \psi(\mathbf{r}, t) \leq \psi + d\psi} d\mathbf{r}} = \frac{\int \delta(\psi - \psi(\mathbf{r}, t)) (\nabla\psi)^2 d\mathbf{r}}{\int \delta(\psi - \psi(\mathbf{r}, t)) d\mathbf{r}}, \quad (87)$$

and using  $\langle \partial\psi/\partial t \rangle = \partial\langle\psi\rangle/\partial t = 0$ , we finally obtain the orbit-averaged-Fokker-Planck equation

$$\frac{\partial}{\partial t}\omega(\psi, t) = \nu(k_0^2\psi - \omega) \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right) + \nu\chi(\psi, t) \frac{\partial}{\partial\psi} \left( \frac{\partial\omega}{\partial\psi} + \beta\gamma\omega \right), \quad (88)$$

$$\Delta\psi = k_0^2\psi - \omega(\psi, t). \quad (89)$$

The steady solution of this equation is the Boltzmann distribution (28).

### 3 The two-species system

#### 3.1 The stochastic equations

For a multi-species system of point vortices, the stochastic equations consistent with the Gibbs canonical distribution at statistical equilibrium are

$$\frac{d\mathbf{r}_i}{dt} = -\frac{1}{\gamma_i} \mathbf{z} \times \nabla_i H - \mu \nabla_i H + \sqrt{2\nu} \mathbf{R}_i(t), \quad (90)$$

where  $H = \sum_{i<j} \gamma_i \gamma_j u(\mathbf{r}_i, \mathbf{r}_j) + (1/2) \sum_{i=1}^N \gamma_i^2 v(\mathbf{r}_i, \mathbf{r}_i)$  is the Hamiltonian. The Fokker-Planck equation determining the evolution of the  $N$ -body distribution function  $P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$  associated with the stochastic equations (90) reads

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \mathbf{V}_i \cdot \frac{\partial P_N}{\partial \mathbf{r}_i} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \cdot \left( \nu \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu P_N \frac{\partial H}{\partial \mathbf{r}_i} \right), \quad (91)$$

where  $\mathbf{V}_i = -(1/\gamma_i) \mathbf{z} \times \partial H / \partial \mathbf{r}_i$  is the total advective velocity of point vortex  $i$ . For  $\nu \neq 0$  and  $\mu \neq 0$ , the stationary solution of the Fokker-Planck equation (91) is the canonical distribution

$$P_N = \frac{1}{Z} e^{-\beta H}, \quad (92)$$

provided that the mobility and the viscosity are related to each other through the Einstein relation (8). The free energy of the  $N$ -body system is given by Eq. (9) and we have the same general properties as for the single species system (see Sec. 2.2). If we restrict ourselves to *positive* temperatures (i.e.  $\mu > 0$ ), the stochastic equations (90) without the rotational term provide a microscopic description of the Debye-Hückel model of electrolytes [34]. At positive temperatures, the full equations (90) also provide a microscopic description of a dissipative guiding center plasma under a strong magnetic field, where like-sign charges repel each other (see Appendix A). Here, we shall be more general and consider both positive and negative temperatures, i.e. repulsive or attractive interactions between like-sign particles.

To be specific, let us consider a globally neutral two-species system made of  $N/2$  point vortices of circulation  $+\gamma$  and  $N/2$  point vortices of circulation  $-\gamma$ . For box-confined vortices interacting through the Newtonian (or Coulombian) potential (56), it can be shown that statistical equilibrium states exist<sup>8</sup> in the canonical ensemble (i.e. the partition function is finite) iff (see [36, 38] and references therein):

$$\beta_c^- \equiv -\frac{16\pi}{N\gamma^2} < \beta < \beta_c^+ = \frac{4\pi}{\gamma^2}. \quad (93)$$

At positive temperatures  $\beta > 0$  (implying  $\mu > 0$ ), the interaction is “repulsive”: the point vortices of the same sign have the tendency to repel each other while the point vortices of opposite sign have the tendency to attract each other. At large temperatures (small  $\beta$ ) a vortex of a given circulation is surrounded by a cloud of vortices with opposite circulation which screen the interaction. This is like the Debye shielding in plasma physics. At small temperatures (large  $\beta$ ) the vortices have the tendency to form pairs or microscopic dipoles  $(+, -)$  similar to “atoms”  $(+e, -e)$  in plasma physics. In this regime, the flow consists in a spatially homogeneous distribution of  $N/2$  individual dipoles. At the critical temperature  $\beta_c^+$ , the most probable state is a gas of  $N/2$  *singular microscopic dipoles*  $(+, -)$  where the vortices of each pair have fallen on each other. This leads to  $N/2$  Dirac peaks made of two vortices of opposite circulation. Thus, the value of the critical temperature  $\beta_c^+$  can be understood by considering the statistical mechanics of only two vortices (one with positive circulation  $+\gamma$  and one with negative circulation  $-\gamma$ ) and determining at which temperature the partition function ceases to be normalizable. The critical temperature  $\beta_c^+$  can thus be obtained from Eq. (57)

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<sup>8</sup>The equation of state is known exactly for a 2D point vortex gas (see, e.g., [35]). Note that the pressure must be positive at positive temperatures  $\beta > 0$  and negative at negative temperatures  $\beta < 0$ . This determines a range of forbidden temperatures where the pressure has not the right sign:  $\beta > \tilde{\beta}_c^+ = 8\pi/\gamma^2$  for neutral systems and  $\beta < \tilde{\beta}_c^- = -8\pi/[(N-1)\gamma^2]$  for a single species system of  $N$  point vortices. According to inequalities (57) and (93), the partition function diverges *strictly before* entering this range of temperatures, so these states are not accessible anyway (see, e.g., [38]).



by taking  $N = 2$  and changing the sign (since the “attraction” of like-sign vortices at  $\beta < 0$  corresponds to the “attraction” of opposite-sign vortices at  $\beta > 0$ ). For  $\beta > \beta_c^+$ , there is no statistical equilibrium state anymore and this regime can be studied dynamically with the stochastic model (90). At negative temperatures  $\beta < 0$  (implying  $\mu < 0$ ), the interaction is “attractive”: the point vortices of the same sign have the tendency to attract each other while the point vortices of opposite sign have the tendency to repel each other. At sufficiently small temperatures (sufficiently negative  $\beta$ ), the vortices have the tendency to form two macroscopic clusters: one cluster of  $N/2$  positive vortices and one cluster of  $N/2$  negative vortices. In this regime, the flow consists in a spatially inhomogeneous distribution of point vortices forming a macroscopic dipole. At the critical temperature  $\beta_c^-$ , the most probable state is a *singular macroscopic dipole*  $(\frac{N}{2}+, \frac{N}{2}-)$  where the vortices of the same species have fallen on each other, creating two Dirac peaks made of  $N/2$  vortices. The value of the critical temperature  $\beta_c^-$  can be understood by considering the statistical mechanics of a single species system of  $N/2$  vortices and determining at which temperature the partition function ceases to be normalizable. The critical temperature  $\beta_c^-$  can thus be obtained from Eq. (57) by making the substitution  $N \rightarrow N/2$ . For  $\beta < \beta_c^-$  there is no statistical equilibrium state anymore and this regime can be studied dynamically with the stochastic model (90).

### 3.2 The BBGKY-like hierarchy

For sake of generality, we first consider a two-species system made of  $N_+ = Nn_+$  point vortices of circulation  $\gamma_+$  (labeled from  $i = 1$  to  $n_+N$ ) and  $N_- = Nn_-$  point vortices of circulation  $\gamma_-$  (labeled from  $i = n_+N + 1$  to  $N$ ). We use the same notations and presentation as in the *equilibrium* study of Pointin & Lundgren [19]. The one-body distributions are defined by

$$P_1^+(\mathbf{r}_1, t) = \int P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t) d\mathbf{r}_2 \dots d\mathbf{r}_N, \quad (94)$$

$$P_1^-(\mathbf{r}_N, t) = \int P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t) d\mathbf{r}_1 \dots d\mathbf{r}_{N-1}. \quad (95)$$

The two-body distributions  $P_2^{++}(\mathbf{r}_1, \mathbf{r}_2, t)$ ,  $P_2^{+-}(\mathbf{r}_1, \mathbf{r}_2, t) = P_2^{-+}(\mathbf{r}_2, \mathbf{r}_1, t)$ ,  $P_2^{--}(\mathbf{r}_1, \mathbf{r}_2, t)$ , and the higher-body distributions are defined similarly. From the Fokker-Planck equation (91), it is easy to derive the complete BBGKY-like hierarchy of the two-species point vortex gas. The general equations are

$$\begin{aligned} \frac{\partial}{\partial t} P_j^{p+, n-}(1, \dots, p+n, t) = & \sum_{i=1}^p \frac{\partial}{\partial \mathbf{r}_i} \left( \nu \frac{\partial P_j^{p+, n-}}{\partial \mathbf{r}_i} + \mu P_j^{p+, n-} \sum_{k=1, k \neq i}^p \gamma_+^2 \frac{\partial' u_{ki}}{\partial \mathbf{r}_i} \right. \\ & + \mu P_j^{p+, n-} \sum_{k=p+1}^j \gamma_+ \gamma_- \frac{\partial' u_{ki}}{\partial \mathbf{r}_i} + \mu N \left( n_+ - \frac{p}{N} \right) \gamma_+^2 \int P_{j+1}^{(p+1)+, n-} \frac{\partial' u_{ki}}{\partial \mathbf{r}_i} d\mathbf{r}_k \\ & \left. + \mu N \left( n_- - \frac{n}{N} \right) \gamma_+ \gamma_- \int P_{j+1}^{p+, (n+1)-} \frac{\partial' u_{ki}}{\partial \mathbf{r}_i} d\mathbf{r}_k + \frac{1}{2} \mu P_j^{p+, n-} \gamma_+^2 \frac{\partial'}{\partial \mathbf{r}_i} v(i, i) \right) + \sum_{i=p+1}^j (+ \leftrightarrow -), \end{aligned} \quad (96)$$

where  $j = p + n$ . The first equation of the BBGKY-like hierarchy for  $P_1^+$  is

$$\begin{aligned} \frac{\partial P_1^+}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_1^+}{\partial \mathbf{r}_1} + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_2^{++}(1, 2) d\mathbf{r}_2 \right. \\ & \left. + \beta N n_- \gamma_+ \gamma_- \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_2^{+-}(1, 2) d\mathbf{r}_2 + \frac{1}{2} \beta P_1^+ \gamma_+^2 \frac{\partial'}{\partial \mathbf{r}_1} v(1, 1) \right]. \end{aligned} \quad (97)$$

The equation for  $P_1^-$  is obtained by interchanging  $+$  and  $-$ . The second equations of the BBGKY-like hierarchy for  $P_2^{++}$  and  $P_2^{+-}$  are

$$\begin{aligned} \frac{\partial P_2^{++}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2^{++}}{\partial \mathbf{r}_1} + \beta \gamma_+^2 P_2^{++}(1, 2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + \beta N \left( n_+ - \frac{2}{N} \right) \gamma_+^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_3^{+++}(1, 2, 3) d\mathbf{r}_3 \right. \\ & \left. + \beta N n_- \gamma_+ \gamma_- \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_3^{++-}(1, 2, 3) d\mathbf{r}_3 + \frac{1}{2} \beta P_2^{++} \gamma_+^2 \frac{\partial'}{\partial \mathbf{r}_1} v(1, 1) \right] + (1 \leftrightarrow 2), \end{aligned} \quad (98)$$

$$\begin{aligned} \frac{\partial P_2^{+-}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2^{+-}}{\partial \mathbf{r}_1} + \beta \gamma_+ \gamma_- P_2^{+-}(1, 2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_3^{+-+}(1, 2, 3) d\mathbf{r}_3 \right. \\ & \left. + \beta N \left( n_- - \frac{1}{N} \right) \gamma_+ \gamma_- \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_3^{+--}(1, 2, 3) d\mathbf{r}_3 + \frac{1}{2} \beta P_2^{+-} \gamma_+^2 \frac{\partial'}{\partial \mathbf{r}_1} v(1, 1) \right] + (1 \leftrightarrow 2). \end{aligned} \quad (99)$$

The equations for  $P_2^{-}$  and  $P_2^{+}$  are obtained by interchanging  $+$  and  $-$ . Next, inserting the Mayer decompositions

$$P_2^{++}(1, 2) = P_1^+(1)P_1^+(2) + P_2'^{++}(1, 2), \quad (100)$$

$$P_2^{+-}(1, 2) = P_1^+(1)P_1^-(2) + P_2'^{+-}(1, 2), \quad (101)$$

in Eq. (97), we obtain the exact equation

$$\begin{aligned} \frac{\partial P_1^+}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_1^+}{\partial \mathbf{r}_1} + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 P_1^+(1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_1^+(2) d\mathbf{r}_2 \right. \\ & + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_2'^{++}(1, 2) d\mathbf{r}_2 + \beta N n_- \gamma_+ \gamma_- P_1^+(1) \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_1^-(2) d\mathbf{r}_2 \\ & \left. + \beta N n_- \gamma_+ \gamma_- \int \frac{\partial' u_{12}}{\partial \mathbf{r}_1} P_2'^{+-}(1, 2) d\mathbf{r}_2 + \frac{1}{2} \beta P_1^+ \gamma_+^2 \frac{\partial'}{\partial \mathbf{r}_1} v(1, 1) \right]. \end{aligned} \quad (102)$$

Inserting the Mayer decompositions

$$\begin{aligned} P_3^{+++}(1, 2, 3) = & P_1^+(1)P_1^+(2)P_1^+(3) + P_1^+(1)P_2'^{++}(2, 3) + P_1^+(2)P_2'^{++}(1, 3) \\ & + P_1^+(3)P_2'^{++}(1, 2) + P_3'^{+++}(1, 2, 3), \end{aligned} \quad (103)$$

$$\begin{aligned} P_3^{+-+}(1, 2, 3) = & P_1^+(1)P_1^+(2)P_1^-(3) + P_1^+(1)P_2'^{+-}(2, 3) + P_1^+(2)P_2'^{+-}(1, 3) \\ & + P_1^-(3)P_2'^{+-}(1, 2) + P_3'^{+-+}(1, 2, 3), \end{aligned} \quad (104)$$

$$\begin{aligned} P_3^{+--}(1, 2, 3) = & P_1^+(1)P_1^-(2)P_1^-(3) + P_1^+(1)P_2'^{-+}(2, 3) + P_1^-(2)P_2'^{++}(1, 3) \\ & + P_1^-(3)P_2'^{+-}(1, 2) + P_3'^{+--}(1, 2, 3), \end{aligned} \quad (105)$$

$$\begin{aligned} P_3^{+--}(1, 2, 3) = & P_1^+(1)P_1^-(2)P_1^-(3) + P_1^+(1)P_2'^{--}(2, 3) + P_1^-(2)P_2'^{+-}(1, 3) \\ & + P_1^-(3)P_2'^{+-}(1, 2) + P_3'^{+--}(1, 2, 3), \end{aligned} \quad (106)$$

in Eqs. (98)-(99), and using Eq. (102) to simplify some terms, we obtain the exact equations

$$\begin{aligned}
\frac{\partial P_2'^{++}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2'^{++}}{\partial \mathbf{r}_1} + \beta \gamma_+^2 P_1^+(1) P_1^+(2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} + \beta \gamma_+^2 P_2'^{++}(1, 2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} \right. \\
& - \beta \gamma_+^2 P_1^+(1) P_1^+(2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 \\
& + \beta N \left( n_+ - \frac{2}{N} \right) \gamma_+^2 P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{++}(2, 3) d\mathbf{r}_3 - \beta \gamma_+^2 P_1^+(2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{++}(1, 3) d\mathbf{r}_3 \\
& + \beta N \left( n_+ - \frac{2}{N} \right) \gamma_+^2 P_2'^{++}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 + \beta N \left( n_+ - \frac{2}{N} \right) \gamma_+^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3'^{+++}(1, 2, 3) d\mathbf{r}_3 \\
& + \beta N n_- \gamma_+ \gamma_- P_2'^{++}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 + \beta N n_- \gamma_+ \gamma_- \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3'^{++-}(1, 2, 3) d\mathbf{r}_3 \\
& \left. + \beta N n_- \gamma_+ \gamma_- P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{+-}(2, 3) d\mathbf{r}_3 + \frac{1}{2} \beta P_2'^{++}(1, 2) \gamma_+^2 \frac{\partial}{\partial \mathbf{r}_1} v(1, 1) \right] + (1 \leftrightarrow 2), \tag{107}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_2'^{+-}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2'^{+-}}{\partial \mathbf{r}_1} + \beta \gamma_+ \gamma_- P_1^+(1) P_1^-(2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} + \beta \gamma_+ \gamma_- P_2'^{+-}(1, 2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} \right. \\
& + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{+-}(2, 3) d\mathbf{r}_3 \\
& + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 P_2'^{+-}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3'^{+-+}(1, 2, 3) d\mathbf{r}_3 \\
& - \beta \gamma_+ \gamma_- P_1^+(1) P_1^-(2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 + \beta N \left( n_- - \frac{1}{N} \right) \gamma_+ \gamma_- P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{--}(2, 3) d\mathbf{r}_3 \\
& - \beta \gamma_+ \gamma_- P_1^-(2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{+-}(1, 3) d\mathbf{r}_3 + \beta N \left( n_- - \frac{1}{N} \right) \gamma_+ \gamma_- P_2'^{+-}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 \\
& \left. + \beta N \left( n_- - \frac{1}{N} \right) \gamma_+ \gamma_- \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_3'^{+--}(1, 2, 3) d\mathbf{r}_3 + \frac{1}{2} \beta P_2'^{+-}(1, 2) \gamma_+^2 \frac{\partial}{\partial \mathbf{r}_1} v(1, 1) \right] + (1 \leftrightarrow 2). \tag{108}
\end{aligned}$$

To order  $1/N$  in the proper thermodynamic limit defined in Sec. 2.2, the foregoing equations reduce to

$$\begin{aligned}
\frac{\partial P_1^+}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_1^+}{\partial \mathbf{r}_1} + \beta N \left( n_+ - \frac{1}{N} \right) \gamma_+^2 P_1^+(1) \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_1^+(2) d\mathbf{r}_2 \right. \\
& + \beta N n_+ \gamma_+^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2'^{++}(1, 2) d\mathbf{r}_2 + \beta N n_- \gamma_+ \gamma_- P_1^+(1) \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_1^-(2) d\mathbf{r}_2 \\
& \left. + \beta N n_- \gamma_+ \gamma_- \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} P_2'^{+-}(1, 2) d\mathbf{r}_2 + \frac{1}{2} \beta P_1^+ \gamma_+^2 \frac{\partial}{\partial \mathbf{r}_1} v(1, 1) \right], \tag{109}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_2'^{++}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2'^{++}}{\partial \mathbf{r}_1} + \beta \gamma_+^2 P_1^+(1) P_1^+(2) \frac{\partial u_{12}}{\partial \mathbf{r}_1} - \beta \gamma_+^2 P_1^+(1) P_1^+(2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 \right. \\
& + \beta N n_+ \gamma_+^2 P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{++}(2, 3) d\mathbf{r}_3 + \beta N n_+ \gamma_+^2 P_2'^{++}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 \\
& \left. + \beta N n_- \gamma_+ \gamma_- P_2'^{++}(1, 2) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 + \beta N n_- \gamma_+ \gamma_- P_1^+(1) \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} P_2'^{+-}(2, 3) d\mathbf{r}_3 \right] + (1 \leftrightarrow 2), \tag{110}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_2'^{+-}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial P_2'^{+-}}{\partial \mathbf{r}_1} + \beta \gamma_+ \gamma_- P_1^+(1) P_1^-(2) \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + \beta N n_+ \gamma_+^2 P_1^+(1) \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_2'^{+-}(2, 3) d\mathbf{r}_3 \right. \\
& + \beta N n_+ \gamma_+^2 P_2'^{+-}(1, 2) \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_1^+(3) d\mathbf{r}_3 - \beta \gamma_+ \gamma_- P_1^+(1) P_1^-(2) \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 \\
& \left. + \beta N n_- \gamma_-^2 P_1^-(1) \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_2'^{--}(2, 3) d\mathbf{r}_3 + \beta N n_- \gamma_-^2 P_2'^{--}(1, 2) \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} P_1^-(3) d\mathbf{r}_3 \right] + (1 \leftrightarrow 2).
\end{aligned} \tag{111}$$

Before considering simplified forms of the above equations, we shall derive useful expressions of the average energy.

### 3.3 The average energy

The average energy

$$E = \langle H \rangle = \sum_{i < j} \int \gamma_i \gamma_j u_{ij} P_N d\mathbf{r}_1 \dots d\mathbf{r}_N + \frac{1}{2} \sum_{i=1}^N \int \gamma_i^2 v_{ii} P_N d\mathbf{r}_1 \dots d\mathbf{r}_N, \tag{112}$$

can be written

$$\begin{aligned}
E = & \frac{1}{2} \gamma_+^2 N_+ (N_+ - 1) \int P_2^{++}(1, 2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 + \frac{1}{2} \gamma_-^2 N_- (N_- - 1) \int P_2^{--}(1, 2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + N_+ N_- \gamma_+ \gamma_- \int P_2^{+-}(1, 2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 + \frac{1}{2} \int [N_+ \gamma_+^2 P_1^+(1) + N_- \gamma_-^2 P_1^-(1)] v(1, 1) d\mathbf{r}_1.
\end{aligned} \tag{113}$$

Substituting the Mayer decomposition in the previous equation, we obtain

$$\begin{aligned}
E = & \frac{1}{2} \gamma_+^2 N^2 n_+ \left( n_+ - \frac{1}{N} \right) \int [P_1^+(1) P_1^+(2) + P_2'^{++}(1, 2)] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + \frac{1}{2} \gamma_-^2 N^2 n_- \left( n_- - \frac{1}{N} \right) \int [P_1^-(1) P_1^-(2) + P_2'^{--}(1, 2)] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + N^2 n_+ n_- \gamma_+ \gamma_- \int [P_1^+(1) P_1^-(2) + P_2'^{+-}(1, 2)] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + \frac{N}{2} \int [n_+ \gamma_+^2 P_1^+(1) + n_- \gamma_-^2 P_1^-(1)] v(1, 1) d\mathbf{r}_1.
\end{aligned} \tag{114}$$

Introducing the average vorticity

$$\omega(\mathbf{r}, t) = N n_+ \gamma_+ P_1^+(\mathbf{r}, t) + N n_- \gamma_- P_1^-(\mathbf{r}, t), \tag{115}$$

and the corresponding stream function (23), the average energy can be put in the form

$$\begin{aligned}
E = & \frac{1}{2} \int \omega \psi d\mathbf{r} - \frac{N}{2} \int [n_+ \gamma_+^2 P_2'^{++}(1, 2) + n_- \gamma_-^2 P_2'^{--}(1, 2)] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + \frac{1}{2} N^2 \int [\gamma_+^2 n_+^2 P_2'^{++}(1, 2) + 2 n_+ n_- \gamma_+ \gamma_- P_2'^{+-}(1, 2) + \gamma_-^2 n_-^2 P_2'^{--}(1, 2)] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& - \frac{1}{2} \gamma_+^2 N n_+ \int P_1^+(1) P_1^+(2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 - \frac{1}{2} \gamma_-^2 N n_- \int P_1^-(1) P_1^-(2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& + \frac{N}{2} \int [n_+ \gamma_+^2 P_1^+(1) + n_- \gamma_-^2 P_1^-(1)] v(1, 1) d\mathbf{r}_1.
\end{aligned} \tag{116}$$

This equation is exact for any  $N$ . At the order  $1/N$  in the proper thermodynamic limit defined in Sec. 2.2, it reduces to

$$\begin{aligned}
E = & \frac{1}{2} \int \omega \psi d\mathbf{r} + \frac{N}{2} \int [n_+ \gamma_+^2 P_1^+(1) + n_- \gamma_-^2 P_1^-(1)] v(1, 1) d\mathbf{r}_1 \\
& + \frac{1}{2} N^2 \int \left[ \gamma_+^2 n_+^2 P_2'^{++}(1, 2) + 2n_+ n_- \gamma_+ \gamma_- P_2'^{+-}(1, 2) + \gamma_-^2 n_-^2 P_2'^{--}(1, 2) \right] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\
& - \frac{1}{2} \gamma_+^2 N n_+ \int P_1^+(1) P_1^+(2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2 - \frac{1}{2} \gamma_-^2 N n_- \int P_1^-(1) P_1^-(2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2. \quad (117)
\end{aligned}$$

### 3.4 The mean field approximation

In the limit  $N \rightarrow +\infty$ , the first equation (109) of the BBGKY-like hierarchy reads

$$\frac{\partial P_1^+}{\partial t} = \nu \frac{\partial}{\partial \mathbf{r}_1} \left\{ \frac{\partial P_1^+}{\partial \mathbf{r}_1} + \beta \gamma_+ P_1^+(1) \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} [N n_+ \gamma_+ P_1^+(2) + N n_- \gamma_- P_1^-(2)] d\mathbf{r}_2 \right\}. \quad (118)$$

Introducing the average vorticity (115) and the corresponding stream function (23), we can rewrite Eq. (118) in the form

$$\frac{\partial P_1^+}{\partial t} + \mathbf{u} \cdot \nabla P_1^+ = \nu \nabla \cdot (\nabla P_1^+ + \beta \gamma_+ P_1^+ \nabla \psi), \quad (119)$$

where  $\mathbf{u} = -\mathbf{z} \times \nabla \psi$  is the mean advective velocity. The equation for  $P_1^-$  is obtained by interchanging  $+$  and  $-$ . Then, defining  $\omega_+ = N n_+ \gamma_+ P_1^+$  and  $\omega_- = N n_- \gamma_- P_1^-$  so that  $\omega = \omega_+ + \omega_-$ , and considering a potential of interaction of the form (56), we obtain the coupled system of equations

$$\frac{\partial \omega_+}{\partial t} + \mathbf{u} \cdot \nabla \omega_+ = \nu \nabla \cdot (\nabla \omega_+ + \beta \gamma_+ \omega_+ \nabla \psi) \equiv -\nabla \cdot \mathbf{J}_+, \quad (120)$$

$$\frac{\partial \omega_-}{\partial t} + \mathbf{u} \cdot \nabla \omega_- = \nu \nabla \cdot (\nabla \omega_- + \beta \gamma_- \omega_- \nabla \psi) \equiv -\nabla \cdot \mathbf{J}_-, \quad (121)$$

$$-\Delta \psi = \omega_+ + \omega_-, \quad (122)$$

with the normalization  $\int \omega_+ d\mathbf{r} = N n_+ \gamma_+ = \Gamma_+$  and  $\int \omega_- d\mathbf{r} = N n_- \gamma_- = \Gamma_-$ . These mean field Fokker-Planck equations conserve the circulations (or vortex numbers) of each species. The steady solutions are the Boltzmann distributions

$$\omega_+ = A_+ e^{-\beta \gamma_+ \psi}, \quad \omega_- = A_- e^{-\beta \gamma_- \psi}. \quad (123)$$

They can also be obtained from the equilibrium BBGKY hierarchy in the limit  $N \rightarrow +\infty$ . Substituting Eq. (123) in Eq. (122), we obtain a mean field equation determining the equilibrium stream function. For a globally neutral system with  $\gamma_+ = -\gamma_- = \gamma$  and  $n_+ = n_- = 1/2$ , this equation reduces, in the symmetric case  $A_+ = -A_-$ , to the celebrated sinh-Poisson equation  $\Delta \psi = 2A \sinh(\beta \gamma \psi)$  studied in [15, 61] (note, however, that there is no fundamental reason why  $A_+ = -A_-$ ; this corresponds to very particular initial conditions). At positive temperatures, and for a globally neutral system, the kinetic equations (120)-(122), without the advective term, are isomorphic to the Debye-Hückel model of electrolytes where like-sign charges repel each other (see Eq. (3) of [34]). Therefore, for  $\beta > 0$ , our study provides a kinetic derivation,

from the microscopic stochastic process (90), of the mean field kinetic equations introduced by Debye & Hückel [34]. At positive temperatures, these equations lead at statistical equilibrium to a spatially uniform state with zero net charge. The novelty of our model is to allow also for the consideration of negative temperatures where like-sign vortices attract each other. For sufficiently negative inverse temperatures, these equations lead to a macroscopic order, typically the formation of a large-scale dipole [62].

The Boltzmann entropy of the two-species point vortex gas

$$S = -N \int [n_+ P_1^+(1) \ln P_1^+(1) + n_- P_1^-(1) \ln P_1^-(1)] d\mathbf{r}_1, \quad (124)$$

can be rewritten (up to an additive constant):

$$S = - \int \left( \frac{\omega_+}{\gamma_+} \ln \frac{\omega_+}{\gamma_+} + \frac{\omega_-}{\gamma_-} \ln \frac{\omega_-}{\gamma_-} \right) d\mathbf{r}. \quad (125)$$

This expression of the Boltzmann entropy can also be obtained from a classical combinatorial analysis [30]. On the other hand, for  $N \rightarrow +\infty$ , the mean field energy is

$$E = \frac{1}{2} \int \omega \psi d\mathbf{r}. \quad (126)$$

Then, the mean field free energy (more properly the Massieu function) reads

$$J = S - \beta E, \quad (127)$$

where  $S$  and  $E$  are given by Eqs. (125) and (126). The mean field free energy (127) can be obtained from Eq. (9) in the limit  $N \rightarrow +\infty$ . It is straightforward to establish that

$$\dot{J} = \int \left( \frac{\mathbf{J}_+^2}{\nu \gamma_+ \omega_+} + \frac{\mathbf{J}_-^2}{\nu \gamma_- \omega_-} \right) d\mathbf{r} \geq 0. \quad (128)$$

This is the appropriate form of  $H$ -theorem in the canonical ensemble:  $\dot{J} \geq 0$  and  $\dot{J} = 0$  iff  $\omega_{\pm}$  are the mean field Boltzmann distributions (123). The free energy  $J[\omega_+, \omega_-]$  is the Lyapunov functional of the Fokker-Planck equations (120)-(122). The Boltzmann distributions (123) are critical points of  $J$  at fixed  $\Gamma_+$  and  $\Gamma_-$  and they are linearly dynamically stable iff they are (local) maxima of  $J$ . If  $J$  is bounded from above, we conclude from Lyapunov's direct method, that the mean field Fokker-Planck equations (120)-(122) will reach, for  $t \rightarrow +\infty$ , a (local) maximum of  $J$  at fixed circulations  $\Gamma_+$  and  $\Gamma_-$ . If several local maxima exist (like dipoles and bars in a periodic domain [63]), the selection of the maximum will depend on a complicated notion of basin of attraction. If there is no maximum of free energy at fixed circulations, the system will have a peculiar behavior and will generate singularities. This is the case in particular for  $\beta < \beta_c^-$  in the situation described in Sec. 3.1. In that case, the partition function diverges and the mean field free energy has no global maximum. This situation can be studied dynamically by solving the mean field Fokker-Planck equations (120)-(122). Preliminary numerical simulations [62] of Eqs. (120)-(122) show the formation of a *singular macroscopic dipole*<sup>9</sup>. The results of these simulations will be reported elsewhere [62]. We note that the relaxation equations (120)-(122) can be obtained from a maximum entropy production principle (MEPP) by maximizing the production of free energy  $\dot{J}$  at fixed circulations  $\Gamma_{\pm}$  (and additional physical constraints) [51].

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<sup>9</sup>Note that a similar growing condensate (dipole) has been studied recently in the context of forced 2D turbulence [64]. Our model of Brownian vortices is physically different from [64] but the connexion between the two studies may be noted.



It is also straightforward to generalize the present results to a multi-species system of Brownian vortices. The general mean field Fokker-Planck equations are given by

$$\frac{\partial \omega_\alpha}{\partial t} + \mathbf{u} \cdot \nabla \omega_\alpha = \nu \nabla \cdot (\nabla \omega_\alpha + \beta \gamma_\alpha \omega_\alpha \nabla \psi), \quad (129)$$

where  $\psi$  is given by Eq. (23) with  $\omega = \sum_\alpha \omega_\alpha$  where  $\omega_\alpha(\mathbf{r}, t) = N_\alpha \gamma_\alpha P_1^{(\alpha)}(\mathbf{r}, t)$  is the ensemble averaged vorticity (proportional to the density) of species  $\alpha$ . The free energy (Massieu function) is  $J = S - \beta E$  where  $S = -\sum_\alpha \int \frac{\omega_\alpha}{\gamma_\alpha} \ln \frac{\omega_\alpha}{\gamma_\alpha} d\mathbf{r}$  and  $E$  is given by Eq. (126). The  $H$ -theorem reads

$$\dot{J} = \nu \sum_\alpha \int \frac{1}{\gamma_\alpha \omega_\alpha} (\nabla \omega_\alpha + \beta \gamma_\alpha \omega_\alpha \nabla \psi)^2 d\mathbf{r}. \quad (130)$$

The relaxation equations (129) have the following properties: (i)  $\Gamma_\alpha = \int \omega_\alpha d\mathbf{r}$  are conserved (ii)  $\dot{J} \geq 0$  (iii)  $\dot{J} = 0$  iff  $\omega_\alpha = A_\alpha e^{-\beta \gamma_\alpha \psi}$  (iv) a steady state is linearly stable iff it is a (local) maximum of  $J$  at fixed  $\Gamma_\alpha$ . Thus, if  $J$  is bounded from above, these equations will relax towards the canonical statistical equilibrium state of the multi-species point vortex gas. We can also extend the approach of Sec. 2.5 to obtain the equations satisfied by the exact vorticity field of each species. They are given by

$$\frac{\partial \omega_\alpha}{\partial t} + \mathbf{u} \cdot \nabla \omega_\alpha = \nu \nabla \cdot (\nabla \omega_\alpha + \beta \gamma_\alpha \omega_\alpha \nabla \psi) + \nabla \cdot (\sqrt{2\nu \gamma_\alpha \omega_\alpha} \mathbf{R}), \quad (131)$$

where  $\psi$  is given by Eq. (42) with  $\omega = \sum_\alpha \omega_\alpha$  where  $\omega_\alpha(\mathbf{r}, t) = \gamma_\alpha \sum_{i \in X_\alpha} \delta(\mathbf{r} - \mathbf{r}_i(t))$  is the exact vorticity field of species  $\alpha$  (here,  $X_\alpha$  denotes the ensemble of point vortices of species  $\alpha$ ). Using the same arguments as those given at the end of Sec. 2.5, we can introduce a coarse-grained vorticity field that smoothes out the Dirac distributions while keeping track of stochastic effects. The evolution of the coarse-grained vorticity is then given by an equation of the form (131) where  $\psi$  is given by Eq. (23) with  $\omega = \sum_\alpha \omega_\alpha$  where  $\omega_\alpha(\mathbf{r}, t)$  denotes now the coarse-grained vorticity of species  $\alpha$ . These stochastic kinetic equations, involving a noise term, could be used to study dynamical phase transitions between different maxima of free energy, like “bars” and “dipoles” in periodic domains. This will be considered in future contributions.

### 3.5 The two-body correlation function

We now consider a globally neutral system such that  $\gamma_+ = +\gamma$ ,  $\gamma_- = -\gamma$  and  $n_+ = n_- = 1/2$ . We also assume that the distribution of point vortices is spatially homogeneous (to leading order) so that  $P_1^\pm(\mathbf{r}, t) = P_0 + \hat{P}^\pm(\mathbf{r}, t)$  where  $\hat{P}^\pm$  is of order  $1/N$ . In that case, the second equations (110)-(111) of the BBGKY hierarchy at the order  $1/N$  reduce to the forms

$$\begin{aligned} \frac{\partial P_2'^{++}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left\{ \frac{\partial P_2'^{++}}{\partial \mathbf{r}_1} + \beta \gamma^2 P_0^2 \frac{\partial' u_{12}}{\partial \mathbf{r}_1} - \beta \gamma^2 P_0^3 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} d\mathbf{r}_3 \right. \\ & \left. + \frac{1}{2} \beta N \gamma^2 P_0 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} [P_2'^{++}(2, 3) - P_2'^{+-}(2, 3)] d\mathbf{r}_3 \right\} + (1 \leftrightarrow 2), \end{aligned} \quad (132)$$

$$\begin{aligned} \frac{\partial P_2'^{+-}}{\partial t} = & \nu \frac{\partial}{\partial \mathbf{r}_1} \left\{ \frac{\partial P_2'^{+-}}{\partial \mathbf{r}_1} - \beta \gamma^2 P_0^2 \frac{\partial' u_{12}}{\partial \mathbf{r}_1} + \beta \gamma^2 P_0^3 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} d\mathbf{r}_3 \right. \\ & \left. + \frac{1}{2} \beta N \gamma^2 P_0 \int \frac{\partial' u_{13}}{\partial \mathbf{r}_1} [P_2'^{-+}(2, 3) - P_2'^{--}(2, 3)] d\mathbf{r}_3 \right\} + (1 \leftrightarrow 2), \end{aligned} \quad (133)$$

and the energy of interaction is

$$E = \frac{N^2\gamma^2}{8} \int \left[ P_2'^{++}(1, 2) - 2P_2'^{+-}(1, 2) + P_2'^{--}(1, 2) \right] u_{12} d\mathbf{r}_1 d\mathbf{r}_2 \\ - \frac{N\gamma^2}{2} \int P_0^2 u_{12} d\mathbf{r}_1 d\mathbf{r}_2 + \frac{1}{2} N\gamma^2 P_0 \int v(1, 1) d\mathbf{r}_1. \quad (134)$$

Considering solutions such that (note the presence of the term  $1/N$  in the definition of  $h$ ):

$$P_2'^{++}(1, 2) = P_2'^{--}(1, 2) = -P_2'^{+-}(1, 2) = -P_2'^{-+}(1, 2) = P_0^2 \left[ h(\mathbf{r}_1 - \mathbf{r}_2, t) + \frac{1}{N} \right], \quad (135)$$

we find that the two-body correlation function  $h(\mathbf{r}_1 - \mathbf{r}_2, t)$  satisfies an equation of the form

$$\frac{\partial h}{\partial t} = 2\nu \frac{\partial}{\partial \mathbf{r}_1} \cdot \left\{ \frac{\partial h}{\partial \mathbf{r}_1} + \beta\gamma^2 \frac{\partial u_{12}}{\partial \mathbf{r}_1} + \beta N\gamma^2 P_0 \int \frac{\partial u_{13}}{\partial \mathbf{r}_1} h(2, 3) d\mathbf{r}_3 \right\}. \quad (136)$$

This returns Eq. (35) for a one component system with a neutralizing background. The correlational energy is

$$\tilde{E} \equiv E - \frac{1}{2} N\gamma^2 P_0 \int v(1, 1) d\mathbf{r}_1 = \frac{N^2\gamma^2}{2} P_0^2 \int h(1, 2) u_{12} d\mathbf{r}_1 d\mathbf{r}_2, \quad (137)$$

which coincides with Eq. (65). Finally, the first equation of the BBGKY-like hierarchy, written at the order  $1/N$ , gives the evolution of the function  $\hat{P}^+(\mathbf{r}, t)$  in the form

$$\frac{\partial \hat{P}^+}{\partial t} = \nu \frac{\partial}{\partial \mathbf{r}_1} \left[ \frac{\partial \hat{P}^+}{\partial \mathbf{r}_1} + \frac{1}{2} \beta N\gamma^2 P_0 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} \left[ \hat{P}^+(2) - \hat{P}^-(2) + 2h(1, 2) P_0 \right] d\mathbf{r}_2 \right. \\ \left. - \beta\gamma^2 P_0^2 \int \frac{\partial u_{12}}{\partial \mathbf{r}_1} d\mathbf{r}_2 + \frac{1}{2} \beta P_0 \gamma^2 \frac{\partial}{\partial \mathbf{r}_1} v(1, 1) \right], \quad (138)$$

and an equivalent equation for  $\hat{P}^-(\mathbf{r}, t)$ . Since we recover the same results as in Sec. 2.4, this two-species model justifies, on a firmer basis, the results obtained in Secs. 2.6 and 2.7 based on a “Jeans-like swindle”. We note that the correlation function (63) and the caloric curve (66) are defined for *all* positive temperatures. In particular, these formulae seem to be valid above the critical inverse temperature  $\beta_c^+$ . This is of course incorrect and this clearly shows that the Debye-Hückel theory is not valid for low temperatures (note that the random phase approximation in [17] also breaks down at low temperatures). In particular, it cannot account for the formation of tightly bound dipoles or pairs  $(+, -)$  as  $\beta \rightarrow \beta_c^+$ . To arrive at the expressions (63) and (66), we have assumed that the three-body correlation function  $P_3'$  could be neglected because it is of order  $1/N^2$ . This is correct “on average” but this is not correct *at any scale*, in particular for small separations. Therefore, it is necessary to take into account higher order correlation functions in the kinetic theory (and go beyond the Debye-Hückel approximation) if we want to reproduce the pair formation at low temperatures. In this respect, we note the important paper of Samaj & Travenec [65] who obtained the exact thermodynamic parameters of the 2D two-components plasma for any value of accessible positive temperatures  $0 \leq \beta \leq \beta_c^+$ . This study clearly shows that the Debye-Hückel theory is only valid at large temperatures.

## 4 Conclusion

In this paper (see also [32]) we have introduced a new model of random walkers in long-range interactions that we called “Brownian vortices”. By definition, their dynamics is governed by stochastic equations of the form (90). This dynamical model is associated with the canonical ensemble and it leads to the Gibbs canonical distribution at statistical equilibrium. In the case of Brownian vortices, we can develop a kinetic theory and analyze all stages of the dynamics. In the mean-field limit, we find that the smooth vorticity field is governed by a parabolic drift-diffusion equation (27) coupled to an equation of the form (23), which is often an elliptic equation like the Poisson equation. This type of equations is actively studied at present by physicists and applied mathematicians in relation to self-gravitating Brownian particles and biological populations experiencing chemotaxis [66]. We think that the present model naturally enters in this line of investigations. In particular, an interesting novelty of this model is to allow for particles (vortices) with positive and negative circulations (while the mass of material particles -like stars or bacteria- is always positive) leading to a wider richness of phenomena.

Another interest of the point vortex model is to allow for both positive and negative temperatures (see Table 2). At negative temperatures, the interaction between like-sign vortices is “attractive” (like gravity) and can create a macroscopic order characterized by the nonuniformity of the spatial distribution of the vortices (typically a monopole or a dipole). The statistical mechanics of point vortices at negative temperatures shares a lot of analogies with the statistical mechanics of stellar systems in astrophysics [35]. It is, furthermore, richer because we can have positive and negative circulations resulting in the separation of a cluster of positive circulations and a cluster of negative circulation (dipole) while gravitational systems typically organize in a single cluster. We have also mentioned the existence of a critical inverse temperature  $\beta_c^-$  below which the partition function diverges and the free energy is not bounded anymore. In that case, we expect the system of point vortices to collapse. It forms a singular monopole  $N\gamma$  (Dirac peak) in the single-species case and a singular dipole  $(\frac{N}{2}\gamma, -\frac{N}{2}\gamma)$  in the two-species case. At positive temperatures, the interaction between like-sign vortices is “repulsive” (like in electrostatics). For a neutral system, this leads to a spatially homogeneous state with correlations which behaves therefore like a 2D Coulombian plasma. We have mentioned the existence of a critical inverse temperature  $\beta_c^+$  above which the partition function diverges and the free energy is not bounded anymore. In that case, we expect the system of point vortices to collapse. It forms a gas of  $N/2$  singular dipoles  $(+\gamma, -\gamma)$  similar to atoms  $(+e, -e)$  in a 2D plasma.

For inhomogeneous systems, the mean field energy (126) scales like  $N^2$ , i.e. the normalized energy  $\epsilon = E/(N^2\gamma^2)$  is of order unity, corresponding to a nonextensive scaling. This regime of negative temperatures, leading to a macroscopic order (like for self-gravitating systems), has been studied by Onsager [24] and others [15, 16, 17, 18, 19, 20, 21, 22, 23] at statistical equilibrium. On the other hand, at positive temperatures, the interaction between like-sign vortices is “repulsive” (like for electric charges) and the physics is very similar to that of a two-dimensional plasma. In that case, we recover the model of Debye-Hückel electrolytes [34]. In the neutral case, the distribution of vortices is spatially homogeneous due to Debye shielding. For homogeneous systems, the correlational energy (137) scales like  $N$  (since  $P'_2 \sim 1/N$ ), so it has an extensive scaling. This regime of positive temperatures, where the usual thermodynamic limit applies, has been studied by Ruelle [67] and others [17, 68] at statistical equilibrium. It was believed for some time that these two approaches (Onsager and Ruelle) were in contradiction. In fact, they correspond to different definitions of the thermodynamic limit describing different situations. The present model of Brownian vortices provides an out-of-equilibrium model that is fully consistent with the results of statistical equilibrium in the canonical ensemble at both positive and negative temperatures in the two situations described above.

The perspectives of this work are the following. On a theoretical point of view, it would be important to study numerically, analytically and with the methods of applied mathematics the mean field Fokker-Planck equations (120)-(122). This has largely been done already in the single species case since the corresponding equations (58)-(59) are isomorphic to the Smoluchowski-Poisson system and Keller-Segel model [32]. However, the two species mean field Fokker-Planck equations (120)-(122) have not been studied extensively so far (especially in the attractive case valid in the regime of negative temperatures). In particular, the formation of the singular macroscopic dipole for  $\beta < \beta_c^-$  is of particular theoretical interest [62]. The study of the  $N$ -body stochastic equations (2) or (90) and the study of the stochastic kinetic equation (55) or (131) are also of interest. When the free energy  $J$  has several local maxima at fixed circulation (metastable states), representing for example monopoles, dipoles, bars or jets like in [63], the stochastic term (noise) in Eq. (55) or (131) can induce *dynamical phase transitions* from one state to the other. It would be interesting to study this phenomenon in detail. On the other hand, on a physical point of view, it would be interesting to know if we can experimentally realize a system of Brownian vortices described by Eqs. (90). It would be interesting to devise experiments (not necessarily in the context of fluid mechanics) where the effective coupling of the particles with a thermal bath can be modeled by stochastic processes of the form (90). In the repulsive case, these equations can describe a dissipative plasma (see [34] and Appendix A). In the attractive case, these equations could describe biological systems where two species of particles organize in two distinct clusters, i.e the particles of the same species attract each other and the particles of different species repel each other.

Let us finally comment on the inequivalence of microcanonical and canonical ensembles for the point vortex system (see also Appendix C). For ordinary point vortices whose dynamics is governed by Hamiltonian equations, the statistical equilibrium state is obtained by maximizing the Boltzmann entropy  $S_B$  at fixed energy  $E$  and circulation  $\Gamma$  (microcanonical ensemble). For Brownian point vortices whose dynamics is governed by Langevin equations, the statistical equilibrium state is obtained by maximizing the Boltzmann free energy  $J_B = S_B - \beta E$  at fixed circulation  $\Gamma$  (canonical ensemble). Canonical stability implies microcanonical stability but the reciprocal is wrong. This means that some states  $(E, \beta)$  that are accessible in the microcanonical ensemble (maxima of entropy  $S$ ) may be inaccessible in the canonical ensemble (they are not maxima of free energy  $J$ ). When this happens, we speak of *ensemble inequivalence*. At equilibrium, ensembles are equivalent for vortices in a disk (or other simple domains) when the Hamiltonian is postulated to be the only constraint [22, 69]. However, they can be inequivalent in other kinds of domains [69] or also in a disk when the conservation of the angular momentum is taken into account [70, 23]. Out-of-equilibrium, the kinetic equations describing Hamiltonian (microcanonical) and Brownian (canonical) point vortices are *very* different in any case. For Hamiltonian vortices, when  $N \rightarrow +\infty$ , the kinetic equations reduce to the 2D Euler equation and finite  $N$  effects are necessary to get the convergence to the microcanonical Boltzmann distribution with temperature  $\beta(E)$  [26]. The relaxation time is larger than  $Nt_D$  and its precise scaling with  $N$  is not known (we are not even sure that the system will reach thermal equilibrium). For Brownian vortices, when  $N \rightarrow +\infty$ , the kinetic equations reduce to the mean-field Fokker-Planck equations (27) or (120)-(122). They converge towards the canonical Boltzmann distribution with the bath temperature  $\beta$  on a diffusive timescale  $L^2/\nu$  independent on  $N$ . Therefore, although the equilibrium states correspond to mean field Boltzmann distributions in the two ensembles, the convergence to these steady states is *radically different* in the Hamiltonian and Brownian descriptions. Thus, our study explicitly shows the fundamental analogies and differences that exist between a microcanonical and a canonical description of point vortices. These analogies and differences are summarized in Table 1 where we compare the kinetic equations of Hamiltonian and Brownian vortices (a completely similar

comparison could be made between Hamiltonian and Brownian systems of material particles with long-range interactions [4, 5, 6, 7, 8]).

## A Dissipative guiding center plasma

The equations of motion of a system of charges in a vertical magnetic field  $\mathbf{B} = B\mathbf{z}$  are

$$m \frac{d\mathbf{v}_i}{dt} = e_i \mathbf{E}_i + e_i \frac{\mathbf{v}_i \times \mathbf{B}}{c}, \quad (139)$$

where  $\mathbf{E}_i = -\nabla_i \Phi = \frac{\partial}{\partial \mathbf{r}_i} \sum_{j \neq i} (2/l) e_j \ln x_{ij}$  is the electric field created self-consistently by the charges [15]. For large fields  $B \gg 1$ , we can neglect the inertial term  $d\mathbf{v}_i/dt \simeq \mathbf{0}$  and make the guiding center approximation

$$\mathbf{v}_i \simeq -\frac{c}{B^2} \mathbf{B} \times \mathbf{E}_i. \quad (140)$$

The equations of motion are then similar to those of a Hamiltonian 2D point vortex gas [15]. For a conservative system (microcanonical ensemble) both positive and negative temperature states are possible [24]. Let us now consider a *dissipative* magnetized plasma so that the equations of motion can be modeled by the stochastic process

$$\frac{d\mathbf{v}_i}{dt} = \frac{e_i}{m} \left( \mathbf{E}_i + \frac{\mathbf{v}_i \times \mathbf{B}}{c} \right) - \xi \mathbf{v}_i + \sqrt{2D} \mathbf{R}_i(t), \quad (141)$$

where  $\xi > 0$  is an effective friction coefficient and  $\mathbf{R}_i(t)$  a random force with  $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$  and  $\langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$ . The Einstein relation reads  $\xi = Dm\beta$ , implying that the temperature is necessarily positive for this model. If we consider both a large field limit  $B \rightarrow +\infty$  and a strong friction limit  $\xi \rightarrow +\infty$  in such a way that  $\xi \sim eB/(mc)$ , we can neglect the inertial term in Eq. (141). Then, after simple manipulations, we obtain the stochastic equation

$$\left( 1 + \frac{m^2 c^2}{e_i^2 B^2} \xi^2 \right) \mathbf{v}_i = \frac{\xi m c^2}{e_i B^2} \mathbf{E}_i - \frac{c}{B^2} \mathbf{B} \times \mathbf{E}_i + \frac{\xi m^2 c^2}{e_i^2 B^2} \sqrt{2D} \mathbf{R}_i - \frac{mc}{e_i B^2} \sqrt{2D} \mathbf{B} \times \mathbf{R}_i. \quad (142)$$

Defining  $\mathbf{Q}_i = a_i \mathbf{R}_i - \mathbf{B} \times \mathbf{R}_i$  with  $a_i = \xi m c / e_i$ , we have  $\langle \mathbf{Q}_i(t) \rangle = \mathbf{0}$  and  $\langle Q_i^\alpha(t) Q_j^\beta(t') \rangle = (a_i^2 + 1) \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$ . Therefore, the stochastic equation (142) can be written in the form of Eq. (90) provided that the parameters are properly reinterpreted. If we consider a dissipative plasma, without imposed magnetic field ( $\mathbf{B} = \mathbf{0}$ ), the stochastic equation (142) obtained in the overdamped limit becomes

$$\frac{d\mathbf{r}_i}{dt} = \frac{e_i}{m\xi} \mathbf{E}_i + \sqrt{\frac{2D}{\xi^2}} \mathbf{R}_i(t). \quad (143)$$

Introducing a mobility  $\mu = 1/(m\xi)$  and a diffusion coefficient  $\nu = D/\xi^2$ , this yields the stochastic equation (90) without the rotational term. This can provide a microscopic model of Debye-Hückel electrolytes in a canonical setting [34] which is valid in  $d = 2$  or  $d = 3$  dimensions.

## B Correlation functions

Considering the correlation function of the exact vorticity field (40), and introducing the one and two-body distributions, we find that

$$\begin{aligned} \langle \omega_d(\mathbf{r}) \omega_d(\mathbf{r}') \rangle &= \langle \gamma^2 \sum_{i,j} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \rangle = \langle \gamma^2 \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}) \rangle \\ &+ \langle \gamma^2 \sum_{i \neq j} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \rangle = N\gamma^2 P_1(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + N(N-1)\gamma^2 P_2(\mathbf{r}, \mathbf{r}'). \end{aligned} \quad (144)$$

We now assume that the system is spatially homogeneous in average, with a vorticity distribution  $\omega = N\gamma P_0$ . Writing  $\delta\omega(\mathbf{r}) = \omega_d(\mathbf{r}) - \omega$ , we get

$$\langle \delta\omega(\mathbf{r}) \delta\omega(\mathbf{r}') \rangle = \langle \omega_d(\mathbf{r}) \omega_d(\mathbf{r}') \rangle - \omega^2. \quad (145)$$

Starting from the identity

$$\langle \omega_d(\mathbf{r}) \omega_d(\mathbf{r}') \rangle = N\gamma^2 P_0 \delta(\mathbf{r} - \mathbf{r}') + N(N-1)\gamma^2 P_2(\mathbf{r}, \mathbf{r}'), \quad (146)$$

and introducing the correlation function  $h(\mathbf{r} - \mathbf{r}')$  through the defining relation  $P_2(\mathbf{r}, \mathbf{r}') = P_0^2(1 + h(\mathbf{r} - \mathbf{r}'))$ , we obtain for  $N \gg 1$ :

$$\langle \delta\omega(\mathbf{r}) \delta\omega(\mathbf{r}') \rangle = \gamma\omega \delta(\mathbf{r} - \mathbf{r}') + \omega^2 h(\mathbf{r} - \mathbf{r}'). \quad (147)$$

The Fourier transform (spectrum) of the vorticity correlations is

$$\langle \delta\hat{\omega}_{\mathbf{k}} \delta\hat{\omega}_{\mathbf{k}'} \rangle = \frac{\gamma\omega}{(2\pi)^2} \left[ 1 + (2\pi)^2 \frac{\omega}{\gamma} \hat{h}(k) \right] \delta(\mathbf{k} + \mathbf{k}'). \quad (148)$$

Finally, using the result (37), we obtain

$$\langle \delta\hat{\omega}_{\mathbf{k}} \delta\hat{\omega}_{\mathbf{k}'} \rangle = \frac{n\gamma^2}{(2\pi)^2} \frac{1}{1 + (2\pi)^2 \beta n \gamma^2 \hat{u}(k)} \delta(\mathbf{k} + \mathbf{k}'). \quad (149)$$

## C A heuristic relaxation equation for point vortices in the microcanonical ensemble

As explained in the Introduction, the dynamical models introduced in this paper correspond to a canonical description of point vortices. In these models, the inverse temperature  $\beta$  is fixed and the energy  $E(t)$  is not conserved. Following the procedure of [44, 25, 45], we can introduce heuristically a dynamical model of point vortices corresponding to a microcanonical description by letting the temperature  $\beta(t)$  depend on time so as to rigorously conserve the energy. Therefore, we now assume in Eq. (2) that  $\mu(t) = \nu\beta(t)$  is a function of time. It is determined by writing  $\dot{E} = \int H \partial_t P_N d\mathbf{r}_1 \dots d\mathbf{r}_N = 0$  and using Eq. (6). This yields

$$\beta(t) = - \frac{\int \sum_{i=1}^N \frac{\partial P_N}{\partial \mathbf{r}_i} \cdot \frac{\partial U}{\partial \mathbf{r}_i} d\mathbf{r}_1 \dots d\mathbf{r}_N}{\gamma^2 \int \sum_{i=1}^N P_N \left( \frac{\partial U}{\partial \mathbf{r}_i} \right)^2 d\mathbf{r}_1 \dots d\mathbf{r}_N}. \quad (150)$$

In the mean field limit  $N \rightarrow +\infty$ , we obtain an equation of the form

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta(t) \gamma \omega \nabla \psi), \quad (151)$$



where  $\beta(t)$  is determined by the condition  $\dot{E} = \int \psi \partial_t \omega \, d\mathbf{r} = 0$  yielding

$$\beta(t) = -\frac{\int \nabla \omega \cdot \nabla \psi \, d\mathbf{r}}{\int \gamma \omega (\nabla \psi)^2 \, d\mathbf{r}}. \quad (152)$$

We can then easily establish (see, e.g., [45]) that

$$\dot{S} = \nu \int \frac{1}{\gamma \omega} (\nabla \omega + \beta \gamma \omega \nabla \psi)^2 \, d\mathbf{r}. \quad (153)$$

The relaxation equation (151) with the constraint (152) has the following properties: (i)  $\Gamma$  and  $E$  are conserved (ii)  $\dot{S} \geq 0$  (iii)  $\dot{S} = 0$  iff  $\omega = A e^{-\beta \gamma \psi}$  (iv) a steady state is linearly dynamically stable iff it is a (local) maximum of  $S$  at fixed circulation and energy. By Lyapunov's direct method, we know that if  $S$  is bounded from above, Eq. (151) will relax towards a (local) maximum of  $S$  at fixed circulation and energy (if several local entropy maxima exist, the choice of the entropy maximum will depend on a notion of basin of attraction). Therefore, this relaxation equation tends to the microcanonical statistical equilibrium. This equation was introduced in [25] by using a maximum entropy production principle (MEPP) and it is a particular case of the general class of nonlinear mean field Fokker-Planck (NFP) equations introduced in [45, 46]. Note that a similar heuristic microcanonical model has been introduced for self-gravitating systems in the form of Kramers and Smoluchowski equations with a time dependent temperature [71, 52]. We stress, however, that Eqs. (151)-(152) do *not* describe the real dynamics of point vortices in the microcanonical ensemble. The exact kinetic theory of point vortices is the one developed in [26] and it leads to very different equations. Therefore, Eqs. (151)-(152) have probably no physical justification. However, they can be used as a numerical algorithm to construct statistical equilibrium states of point vortices in the microcanonical ensemble (note in this respect that it is possible to generalize the MEPP so as to include the conservation of angular momentum in a disk or in an infinite domain). Similarly, Eq. (27) can be used as a numerical algorithm to construct statistical equilibrium states in the canonical ensemble. These remarks give them some practical interest. Note also that the dynamical evolution of the microcanonical equation (151) with the constraint (152) is very different from the dynamical evolution of the canonical equation (27) with fixed  $\beta$ . Indeed, in the microcanonical ensemble, there exists a statistical equilibrium state for any value of the energy  $E$  so that Eq. (151) with Eq. (152) will relax towards this equilibrium state. By contrast, in the canonical ensemble, there is no equilibrium state for  $\beta < \beta_c$  and, in that case, the relaxation equation (27) leads to vortex collapse.

For the multi-species system, the heuristic microcanonical model in the mean field limit is

$$\frac{\partial \omega_\alpha}{\partial t} + \mathbf{u} \cdot \nabla \omega_\alpha = \nu \nabla \cdot (\nabla \omega_\alpha + \beta(t) \gamma_\alpha \omega_\alpha \nabla \psi), \quad (154)$$

where  $\beta(t)$  is determined by the condition  $\dot{E} = \int \psi \partial_t \omega \, d\mathbf{r} = 0$  yielding

$$\beta(t) = -\frac{\int \nabla \omega \cdot \nabla \psi \, d\mathbf{r}}{\int \omega_2 (\nabla \psi)^2 \, d\mathbf{r}}, \quad (155)$$

where  $\omega = \sum_\alpha \omega_\alpha$  and  $\omega_2 = \sum_\alpha \gamma_\alpha \omega_\alpha$ . We can easily establish that

$$\dot{S} = \nu \sum_\alpha \int \frac{1}{\gamma_\alpha \omega_\alpha} (\nabla \omega_\alpha + \beta \gamma_\alpha \omega_\alpha \nabla \psi)^2 \, d\mathbf{r}. \quad (156)$$

The relaxation equations (154) with the constraint (155) have the following properties: (i)  $\Gamma_\alpha = \int \omega_\alpha \, d\mathbf{r}$  and  $E$  are conserved (ii)  $\dot{S} \geq 0$  (iii)  $\dot{S} = 0$  iff  $\omega_\alpha = A_\alpha e^{-\beta \gamma_\alpha \psi}$  (iv) a steady state is

linearly dynamically stable iff it is a (local) maximum of  $S$  at fixed circulations  $\Gamma_\alpha$  and energy. Therefore, if  $S$  is bounded from above, these equations will relax towards the microcanonical statistical equilibrium state of the multi-species point vortex gas. These equations can also be derived from the maximum entropy production principle [51].

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Domain of validity	Hamiltonian	Brownian
Microscopic $N$ -body model for $\mathbf{r}_i(t)$	Deterministic Kirchhoff equations (see Eq. (2) of [26])	Stochastic Langevin equations (see Eq. (2))
Exact kinetic equation for $P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$	Liouville equation (see Eq. (3) of [26])	$N$ -body Fokker-Planck equation (see Eq. (6))
Exact kinetic equation for $\omega_d(\mathbf{r}, t) = \sum_{i=1}^N \gamma \delta(\mathbf{r} - \mathbf{r}_i(t))$	Klimontovich equation (see Eq. (69) of [26])	Stochastic kinetic equation (see Eq. (42))
Kinetic equation for $\omega(\mathbf{r}, t) = N\gamma P_1(\mathbf{r}, t)$ when $N \rightarrow +\infty$	Euler-Poisson equation (see Eq. (35) of [26])	Fokker-Planck-Poisson equation (see Eq. (25))
Violent relaxation of the 2D Euler equation. Kinetic equations for the coarse-grained vorticity $\bar{\omega}(\mathbf{r}, t)$	Phenomenological equations obtained from the MEPP (see Eq. (15) of [44] and Eqs. (23)-(30) of [72]) or kinetic equation obtained from a quasilinear theory (see Eq. (18) of [73])	
Diffusion coefficient for the violent relaxation	See Eq. (8) of [74], Eq. (4.9) of [71] or Eq. (27) of [73]	
Integro-differential kinetic equation for $\omega(\mathbf{r}, t)$ at the order $O(1/N)$	See Eq. (54) of [26] (collective effects neglected)	unknown
Kinetic equation at the order $O(1/N)$ for axisymmetric flows	See Eq. (11) of [28] (with collective effects) and Eq. (48) of [26] (collective effects neglected)	unknown
Test particle in a (thermal) bath	Fokker-Planck equation with a <i>fixed</i> potential. See Eq. (141) of [26] or Eq. (140) of [26] (thermal)	
Diffusion coefficient for a test particle in a (thermal) bath	See Eq. (102) of [26] or Eq. (136) of [26] (thermal)	

Table 1: Summary of the different kinetic equations for Hamiltonian and Brownian vortices. A similar Table would apply to Hamiltonian and Brownian systems of material particles with long-range interactions [4, 5, 6, 7, 8].

	$\beta < \beta_c^-$	$\beta < 0$	$\beta > 0$	$\beta > \beta_c^+$
2D self-gravitating Brownian particles			$(m, m)$ attract $\Rightarrow$ cluster	Collapse: $\Rightarrow$ Dirac peak of mass $Nm$
2D neutral plasma			$(+e, +e)$ repel $(-e, -e)$ repel $(+e, -e)$ attract $\Rightarrow$ homogeneous state	Collapse: $N/2$ singular “atoms” $(+e, -e)$
Single species 2D Brownian vortices	Collapse: $\Rightarrow$ Dirac peak of circulation $N\gamma$	$(\gamma, \gamma)$ attract $\Rightarrow$ cluster (monopole)	$(\gamma, \gamma)$ repel $\Rightarrow$ concentration at the boundary	
Multi species 2D Brownian vortices	Collapse: singular macroscopic dipole $(+\frac{N}{2}\gamma, -\frac{N}{2}\gamma)$	$(+\gamma, +\gamma)$ attract $(-\gamma, -\gamma)$ attract $(+\gamma, -\gamma)$ repel $\Rightarrow$ two clusters of opposite sign (dipole)	$(+\gamma, +\gamma)$ repel $(-\gamma, -\gamma)$ repel $(+\gamma, -\gamma)$ attract $\Rightarrow$ homogeneous state	Collapse: $N/2$ singular microscopic dipoles $(+\gamma, -\gamma)$

Table 2: Schematic “phase diagram” showing the analogies between 2D self-gravitating systems (or bacterial populations [32]), 2D Coulombian plasmas and 2D point vortices in the canonical ensemble [38]. For a system of  $N$  self-gravitating Brownian particles with mass  $m$ :  $\beta_c^+ = 4/(GMm)$ . For a neutral plasma with  $N/2$  charges  $+e$  and  $N/2$  charges  $-e$ :  $\beta_c^+ = 2/e^2$ . For a system of  $N$  Brownian point vortices with circulation  $\gamma$ :  $\beta_c^- = -8\pi/(N\gamma^2)$ . For a multi-species system of Brownian point vortices with  $N/2$  vortices  $+\gamma$  and  $N/2$  vortices  $-\gamma$ :  $\beta_c^- = -16\pi/(N\gamma^2)$  and  $\beta_c^+ = 4\pi/\gamma^2$ . In the microcanonical ensemble,  $\beta(E)$  is a function of the energy. There is an equilibrium state for each value of the energy  $E$ , so there is no collapse. The critical temperatures  $\beta_c^\pm$  correspond to  $E \rightarrow \mp\infty$  so the states  $\beta < \beta_c^-$  and  $\beta > \beta_c^+$  are not accessible in the microcanonical ensemble.